Compromise, Don’t Optimize: A Prior-Free Alternative to Perfect Bayesian Equilibrium

KARL SCHLAG AND ANDRIY ZAPECHELNYUK

ABSTRACT. Perfect Bayesian equilibrium is the classic solution concept for games with incomplete information, where players optimize under given beliefs over states. We introduce a new concept called perfect compromise equilibrium, where players find compromise decisions that are good in all states. This solution concept is tractable even if states are high dimensional as it does not rely on priors, and it always exists. We demonstrate the power of our solution concept in prominent economic examples, including Cournot and Bertrand markets, Spence’s signaling, and bilateral trade with common value.

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Schlag: Department of Economics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria. E-mail: karl.schlag@univie.ac.at.
Zapechelnyuk: School of Economics and Finance, University of St Andrews, Castlecliffe, the Scores, St Andrews KY16 9AR, UK. E-mail: az48@st-andrews.ac.uk.
1. Introduction

The classic solution concept in games of incomplete information is perfect Bayesian equilibrium (PBE). However, many aspects surrounding this concept raise doubts as to whether this is the way we should solve such games. Players are uncertain about the primitives and yet they are willing to add even more details by assigning probabilities to possible events. PBE are often difficult to compute or are intractable, thus confining many of our insights to be limited to very simple examples with two types.

We set out to introduce a new concept for solving games with informational uncertainty, where players do not know something about the environment or about what others know. On the one hand, the concept should avoid the above pitfalls. It should not be based on a specific prior over uncertain events. It should allow different players to see the world differently. It should be tractable in salient examples. On the other hand, the concept should not be too different from PBE. Preferences of players should be comparable to those used in PBE. In particular, known random events, such as tossing a coin, should be treated as in PBE. There should be common knowledge of the equilibrium strategies. So there will be strategic certainty. Challenges are multifold, such as avoiding dynamic inconsistency, modeling learning in the game, and holding off the criticism of those who would like us to choose a different model of optimization under uncertainty. Our solution includes a new rhetoric based on balancing losses and finding compromises. Our examples reveal the tractability of our concept. Our conviction is to provide an alternative to PBE which is reasonable and insightful in applications.

The key ingredient to our approach is that players do not optimize, they compromise. Traditionally, players optimize with respect to some beliefs. However, this is not possible in our model as we do not assume a prior over uncertain events. Instead, players compromise. They look in each possible event and the loss of not playing a best response in this event, and then make a choice that balances these losses across all events.

Our new solution concept for games with incomplete information is called perfect compromise equilibrium (short, PCE). Uncertainty is captured by including a set of states (or events). At the outset, nature chooses one of these states, but none of the players knows which state has been chosen. Each later information set is assigned a subset of states, those that are conceivable for the player who moves at that information set. Information about the state might be revealed to some of the players during the game, like a buyer learning her value. This is modeled by a move of player 0 who is in charge of all chance moves. The mixed strategy of player 0 is common knowledge. The rest of the game unfolds like a standard extensive form game. For each information set, there is a set of actions of the player who is making the choice, a set of conceivable states, and a belief, conditional on a conceivable state, over the decision nodes within this information set. These elements allow us to compute expected payoffs at each information set conditional on each conceivable state using Bayes’ rule. The performance of an action at an
information set is evaluated for a conceivable state by comparing its payoff to the best payoff that could have been attained if one had known the state and had been allowed to choose a different action at this information set. We refer to this difference as the \textit{loss} at that information set. Actions are evaluated by their \textit{maximum loss} across all conceivable states. An action is called a \textit{best compromise} if no other action at this information set has a lower maximum loss.

Analogous to sequential rationality in PBE, we impose a minimal consistency requirement on the conceivable states. A state that is conceivable at an information set must remain conceivable at all information sets that can be reached under this state given the strategy profile and the beliefs.

The elimination of priors over uncertain events comes with numerous advantages in comparison to PBE. Solutions are often easier to obtain. They are more parsimonious as they do not change with a prior. Solutions are more intuitive as they are simple and depend on concrete model primitives. Preferences condition on observables and not on fictitious distributions. Solutions can be justified in front of others who have different priors. Uncertainty and ambiguity enter the model distinctly.

We do not introduce a new non-expected utility method for evaluating strategies. The evaluation of a strategy under a given state is as under PBE. The term “compromise” reflects the fact that the equilibrium strategy will be evaluated across all states, regardless of which state has been realized. The adjective “perfect” refers to the fact that optimization takes place at all information sets, not only at the outset.

Learning about the uncertainty is modeled by updating conceivable sets. Dynamic inconsistency that potentially arises by a player incorporating different worst cases at different information sets is avoided by imposing deviations only at the information set where a strategy is being evaluated. Future choices are not questioned at that point as the player anticipates that she will reoptimize at each future information set.

Our concept has nice properties. It allows for addressing problems with rich uncertainty that are intractable under PBE. It collapses to PBE is there is only a single state. It always exists under standard assumptions. Players will make rational moves as strictly dominated strategies will not be chosen.

Four examples are presented to show how the PCE concept applies. We first consider common uncertainty and investigate Cournot competition with unknown demand. Demand is a state drawn by nature. Firms only know that it is bounded by two given linear functions. Next, we consider private information and analyze Bertrand competition where each firm only knows its own marginal cost. We then move on to signaling and look at Spence’s job market. Firms only know that the cost-productivity combinations of the worker are bounded by two given linear functions. Finally, we consider asymmetric information with common value and analyze sequential bilateral trade where one side has private information.
We are proud to highlight some aspects of our findings in these examples in terms of realism, good compromises, and new insights. All our examples are more realistic than those found in the literature, as we do not have to confine ourselves to parametric models of uncertainty and to models with two states (high and low). Our examples are concerned with strategic decision making under rich uncertainty where states are high dimensional and PBE analysis is no longer tractable. Compromise values are negligible in the Cournot and Bertrand competition settings. In these contexts it makes little sense to think in more detail about which state is really the true one, as payoffs would only be slightly higher in some states but could be substantially lower in other states. New insights appear. We find that adding uncertainty makes firms more competitive under Cournot competition and less competitive under Bertrand competition. In the separating equilibrium of Spence’s job market signaling game, better educated workers are not necessarily more productive, unlike in the classic model with two types. In the sequential bilateral trade with common value, we find that trade is possible, as opposed to the famous no-trade theorem for PBE in this context (Milgrom and Stokey, 1982). Under PCE the possibility that the trading partners have different valuations leads to trade with positive probability, as ignoring this possibility generates losses that the traders want to minimize. Under PBE there is no trade as the trading partners always agree on the expected valuation of the good.

**Related Literature.** Our paper contributes to the literature on robustness and ambiguity in games of incomplete information.

The most related paper to ours is Hanany, Klibanoff and Mukerji (2018). They consider a game of incomplete information in which traditional players are replaced by players with smooth ambiguous preferences, as introduced in Klibanoff, Marinacci and Mukerji (2005). Hanany, Klibanoff and Mukerji (2018) show how to update information in a dynamically consistent fashion, and this updating satisfies the one shot deviation principle. Hence, their approach is better founded in the axiomatic context. However, the intricacies that emerge from their mathematical formulation make their solution concept complex, and impede finding tractable solution in examples with rich state spaces, like the ones in this paper.

An important ingredient of our solution concept is our use of compromise for making choices when the true state is unknown. A popular alternative approach in the literature on ambiguity is maximin preferences (Wald, 1950; Gilboa and Schmeidler, 1989). These preferences have been brought to simultaneous-move games with incomplete information and multiple priors by Epstein and Wang (1996), Kajii and Morris (1997), Kajii and Ui (2005), and Azrieli and Tepner (2011). While this approach can be suitable in many applications, it leads to unintuitive results in our examples. To obtain nontrivial results, additional structural assumptions need to be added, such as assuming knowledge of the mean state, which reintroduces the priors that we are trying to eliminate.

Our idea of best compromise has origins in minimax regret (Savage, 1951) and connects to \( \varepsilon \) optimality. Our optimization criterion differs from minimax regret
as evaluation occurs at each information set, while minimax regret traditionally evaluates regret ex post. For an investigation of minimax regret under strategic uncertainty see Linhart and Radner (1989), and under partial strategic uncertainty see Renou and Schlag (2010).

PCE can be considered as a generalization of ex post Nash equilibrium (Cremer and McLean, 1985). It can be thought of as an $\varepsilon$-ex post Nash equilibrium in which the smallest possible value of $\varepsilon$ is chosen for each player. As a concept, $\varepsilon$-Nash equilibrium (Radner, 1980) is usually seen as a play under the restriction that deviations are only undergone if payoff improvements are substantial. Our interpretation is different. The value of $\varepsilon$ does not measure the inertia that needs to be overcome, but instead it measures the compromise needed to accommodate all possible states. In particular, the threshold $\varepsilon$ is endogenous in a PCE.

Stauber (2011) analyzes the local robustness of PBE to small degrees of ambiguity about player’s beliefs. In particular, it does investigate how the players adjust their play to this ambiguity, unlike our paper.

In fact, PCE can be interpreted as a globally robust version of PBE where robustness (Huber, 1965) means to make choices that also perform well if the model is slightly misspecified. Being a compromise, our suggested strategies perform well in each state given how others make their choices, never doing too badly relative to what could be achieved in that state. This stands in contrast to the maximin utility approach that focuses attention on the state where payoffs are lowest.

We proceed as follows. In Section 2 we introduce our solution concept. In Section 3 we illustrate PCE in four self-contained examples. Section 4 concludes. All proofs are in Appendix A. Some additional examples are in Appendix B.

2. Perfect Compromise Equilibrium

We introduce a solution concept called perfect compromise equilibrium (PCE). The essential difference of PCE from perfect Bayesian equilibrium (PBE) is as follows. In PBE, players choose strategies that are best under their beliefs about an uncertain state. Instead, in PCE players choose compromises that are good in all realizations of the state. When a player makes a move, she evaluates her action in each state by her loss relative to the best payoff in that state, and then finds a best compromise action that achieves the lowest maximum loss.

A formal definition of PCE is presented in Section 2.1 below. A reader who wishes to be spared with the formalities and seeks to understand the essence of PCE and its applicability can jump to Section 3 that presents self-contained examples.

2.1. Formal Setup. Consider a finite extensive-form game described by $(N_0, \mathcal{G}, u, s_0)$, where $N_0 = \{0, 1, \ldots, n\}$ is a set of players, $\mathcal{G}$ is a finite game tree, $u$ is a profile of payoff functions, and $s_0$ is a strategy of player 0 who is nonstrategic.

Game tree $\mathcal{G}$ describes the order of players’ moves, players’ information sets, and actions that are available at each information set. It is defined by a set of linked
decision nodes and terminal nodes that form a tree. Each decision node is assigned three elements: a player $i$, an information set $\phi$ that contains this decision node, possibly, together with other decision nodes that player $i$ cannot distinguish, and a set of actions available to player $i$ at that information set. Information sets and action sets satisfy the standard assumptions of games with perfect recall. Let $\phi_0$ be the initial decision node of the game, let $\Phi$ be the set of all information sets except $\phi_0$, and let $\mathcal{T}$ be the set of terminal nodes of the game. Let $i(\phi)$ be the player that makes a move at an information set $\phi$, and let $A(\phi)$ be a finite set of actions available at $\phi$.

The game starts with a move of nature. Nature moves only once, at the initial decision node $\phi_0$. An action of nature $\omega$ is called state and is chosen from a finite set $\Omega$ of available states, so $\Omega = A(\phi_0)$.

The game terminates after finitely many moves at some terminal node, and players obtain payoffs. A payoff function of each player $i \in N_0$ specifies the payoff $u_i(\tau)$ of player $i$ at each terminal node $\tau \in \mathcal{T}$. Player 0 is nonstrategic, so we assume that her payoff is always zero.

A strategy of each player $i \in N_0$ prescribes a mixed action $s_i(\phi)$ for each information set $\phi \in \Phi$ in which $i$ makes a move, so $i(\phi) = i$ and $s_i(\phi) \in \Delta(A(\phi))$. Player 0 is non-strategic and follows an exogenously given strategy $s_0$. A strategy profile $s$ describes the behavior of all players throughout the game, so $s(\phi) \in \Delta(A(\phi))$ for each $\phi \in \Phi$ and each state $\omega \in \Omega$.

Like in Bayesian games, we specify not only strategies, but also beliefs of the players. The crucial difference is that, in our setting, the players do not form priors about the move of nature, that is, about states in $\Omega$. Instead, ex ante they consider all states as possible. A player can rule out the possibility of some states by being in an information set which cannot be reached under these states. Thus, the belief in each information set is decomposed into two elements: conceivable set and posterior. A conceivable set deals with the player’s ambiguity about the state. A posterior deals with the uncertainty that arises through probabilistic moves of other players at earlier information sets.

A state $\omega$ is called conceivable at an information set $\phi \in \Phi$ if, upon reaching $\phi$, the possibility that the true state is $\omega$ is not ruled out. A conceivable set $B(\phi)$ is a set of states that are conceivable at $\phi$. Formally, for each information set $\phi \in \{\phi_0\} \cup \Phi$, $B(\phi)$ is a nonempty subset of $\Omega$, with the convention that all states are initially conceivable, so $B(\phi_0) = \Omega$.

A posterior $\beta(\phi|\omega)$ assigns to each information set $\phi \in \Phi$ and each state $\omega \in B(\phi)$ a probability distribution over decision nodes in $\phi$ conditional on the state being $\omega$.

Like in PBE, we will require consistency of beliefs.

**Definition 1.** A profile $(B, \beta)$ of conceivable sets and posteriors is consistent with a strategy profile $s$ if the following conditions hold for all $\omega \in \Omega$.

(a) Let $\phi' \in \Phi$. If there does not exist a path in $\mathcal{G}$ from $\phi_0$ to $\phi'$ in which nature’s move at $\phi_0$ is $\omega$, then $\omega \notin B(\phi')$. 
(b) Let $\phi \in \Phi$ and let $\alpha$ be a decision node in $\phi$ such that $\beta(\phi|\omega)(\alpha) > 0$. Let $\phi' \in \Phi$ be an information set that is reached from node $\alpha$ in one move under $s$ with a strictly positive probability. Then $\omega \in B(\phi')$ and $\beta(\phi|\omega)$ follows Bayes’ rule.

Condition (a) stipulates which states must be excluded from the conceivable set. Every state under which the current information set cannot be reached no matter what actions players choose must be ruled out. Condition (b) stipulates which states must be included in the conceivable set and how the posteriors are computed. Every state under which the current information set is reachable from an earlier information set under a given strategy profile is conceivable and cannot be ruled out. The posteriors follows Bayes’ rule whenever possible given the posteriors in the earlier information sets.

We now define PCE. Consider a profile of strategies, conceivable sets, and posteriors $(s, B, \beta)$. Let $\bar{u}_i(x_i, s|\omega, \phi, \beta)$ denote the expected payoff of player $i$ from choosing a mixed action $x_i \in \Delta(A(\phi))$ in an information set $\phi$ where $i$ makes her move, so $i(\phi) = i$, conditional on the state being $\omega \in B(\phi)$ and assuming that the play is given by $s$ elsewhere in the game. The payoff difference

$$\sup_{a_i \in A(\phi)} \bar{u}_i(a_i, s|\omega, \phi, \beta) - \bar{u}_i(x_i, s|\omega, \phi, \beta)$$

is called player $i$’s loss from choosing mixed action $x_i$ at information set $\phi$ when the state is $\omega$. It describes how much better off player $i$ could have been at this information set under state $\omega$ if, instead of choosing $x_i$, she had chosen the best action in this state, without changing actions prescribed by $s$ in any other information set. The maximum loss of player $i$ from choosing mixed action $x_i$ in information set $\phi$ where $i = i(\phi)$ is given by

$$l(x_i, s|\phi, \beta) = \sup_{\omega \in B(\phi)} \left( \sup_{a_i \in A(\phi)} \bar{u}_i(a_i, s|\phi, \omega, \beta) - \bar{u}_i(x_i, s|\phi, \omega, \beta) \right).$$

So the maximum (supremum) is sought over all states that are conceivable for player $i$ at $\phi$.

Our equilibrium concept requires strategies to be chosen optimally in the sense of minimizing the players’ maximum losses given their beliefs, and the beliefs to be consistent given the strategies.

**Definition 2.** A profile $(s, B, \beta)$ is called a perfect compromise equilibrium if

(a) each player chooses a best compromise in each of her information sets, so for each $\phi \in \Phi$, strategy $s_i$ of player $i = i(\phi)$ minimizes her maximum loss at $\phi$:

$$s_i(\phi) \in \arg\min_{x_i \in \Delta(A(\phi))} l(x_i, s|\phi, \beta).$$

(b) profile $(B, \beta)$ of conceivable sets and posteriors is consistent with strategy profile $s$.

**Remark 1.** In some applications, it is unrealistic to assume that players can choose mixed actions. Our definition of PCE can be easily adjusted if players are
only allowed to use pure actions. In this case, each player minimizes her maximal loss among her pure actions, so instead of (1) we require

\[ s_i(\phi) \in \arg \min_{a_i \in A(\phi)} l(a_i, s|\phi, \beta). \]  

(1')

2.2. Properties of Perfect Compromise Equilibrium. Let us mention a few properties of PCE.

First, we establish existence of PCE.

**Theorem 1.** A perfect compromise equilibrium exists.

The proof is in Appendix A.1.

Second, PCE is equivalent to PBE when there is certainty about the state, so the game is of complete information. In such a game, where the set of states \( \Omega \) is a singleton, conceivable sets are all singletons. So, an action minimizes the maximum loss of a player if and only if it is a best response.

Third, an ex post Nash equilibrium (if it exists) is a PCE. Here, the term *ex post* refers to the game in which the state (or move of nature) is observed by all. In an ex post Nash equilibrium, regardless of the realized state, each player’s strategy in each information set is a best response. Thus, the maximum loss in each information set is zero. No other strategy can further reduce this loss. So, this is a PCE in which all players have zero losses.

Finally, the concept of PCE respects dominance and iterated dominance. We say that an action \( a_i \in A(\phi) \) at an information set \( \phi \) is strictly dominated for player \( i = i(\phi) \) if there exists a mixed action \( x_i \in \Delta(A(\phi)) \) such that player \( i \)'s payoff from choosing \( a_i \) is strictly worse than that from choosing \( x_i \), regardless of the state \( \omega \in \Omega \) and of the choices of other players at any of their information sets. Iterated dominance is defined in a standard way: after having excluded actions that were strictly dominated in previous rounds, one checks the dominance condition w.r.t. the remaining actions of each player. Observe that if an action \( a_i \) is strictly dominated, then it cannot be a best compromise at the correspondent information set, and thus it cannot be a part of any PCE. This argument can be iterated, so any iterated strictly dominated action cannot be a part of any PCE.

3. Examples

We illustrate our solution concept in a few applications that are prominent in the literature. We consider Cournot and Bertrand duopoly, Spence’s job market signaling, and bilateral trade. Moreover, in Appendix B we analyze a forecasting problem and a public good game.

In these applications, actions traditionally belong to an interval or to the positive reals. The concept of PCE is easily extended to allow for infinite sets. Alternatively, one can discretize the sets of actions and states.

Traditionally, uncertainty is incorporated in such models in a very simple fashion, often only considering two states, high and low. We consider richer (in some
cases, infinite dimensional) sets of uncertain events in order to capture more realistic uncertainty.

3.1. Cournot Duopoly with Unknown Demand. We investigate how two firms compete in quantities when neither firm knows the demand.

There are two firms that produce a homogeneous good. For clarity of exposition, we assume that there are no costs of production. Each firm $i = 1, 2$ chooses a number of units $q_i \geq 0$ to produce. Choices are made simultaneously. The firms face an inverse demand function given by $P(q_1 + q_2)$. Each firm $i$’s profit is given by

$$u_i(q_i, q_{-i}; P) = P(q_i + q_{-i})q_i, \quad i = 1, 2.$$  

Neither firm knows the inverse demand $P$, they only know a set that contains $P$.

Let

$$\bar{P}(q) = \bar{a} - \bar{b}q \quad \text{and} \quad \bar{P}(q) = \bar{a} - \bar{b}q,$$

where $\bar{a} \geq \bar{a} > 0$ and $\bar{a}/\bar{b} \geq a/b > 0$.

We assume that $P$ belongs to the set $\mathcal{P}$ of inverse demand functions that satisfy

$$P(q) \leq P(q) \leq \bar{P}(q) \quad \text{and} \quad P'(q) \leq P'(q). \tag{2}$$

A firm $i$’s maximum loss of choosing quantity $q_i$ when the other firm chooses $q_{-i}$ is given by

$$l_i(q_i, q_{-i}) = \sup_{P \in \mathcal{P}} \left( \sup_{q_i' \geq 0} u_i(q_i', q_{-i}; P) - u_i(q_i, q_{-i}; P) \right).$$

The maximum loss describes how much more profit firm $i$ could have obtained if it had known the demand $P$ when anticipating the other firm to produce $q_{-i}$. Firm $i$’s best compromise is a quantity $q_i^*$ that achieves the lowest maximum loss for a given choice $q_{-i}$ of the other firm:

$$q_i^* \in \arg \min_{q_i \geq 0} l_i(q_i, q_{-i}).$$

A strategy profile $(q_1^*, q_2^*)$ is a perfect compromise equilibrium if each firm chooses a best compromise given the choice of the other firm.

This application can be embedded in our formal setting as described in Section 2. As the demand $P$ is unknown, it is identified as the state. So, the set of states is $\mathcal{P}$. In the formal game, first nature chooses the state, and then two firms simultaneously choose their quantities without observing the state. Conceivable sets and posteriors are trivial, as this is a simultaneous move game with no private information.

**Proposition 1.** There exists a unique perfect compromise equilibrium whose strategy profile $(q_1^*, q_2^*)$ is given by

$$q_i^* = \frac{1}{3\left(\sqrt{b} + \sqrt{\bar{b}}\right)} \left(\frac{a}{\sqrt{b}} + \frac{\bar{a}}{\sqrt{\bar{b}}}\right), \quad i = 1, 2. \tag{3}$$
The associated maximum losses are
\[ l_i(q^*_i, q^*_{-i}) = \frac{(\bar{a} - \bar{b})^2}{4\bar{b}\left(\sqrt{\bar{b}} + \sqrt{\bar{b}}\right)^2}, \quad i = 1, 2. \]  
(4)

The proof is in Appendix A.2.

Let us discuss the strategic concerns underlying the PCE in this game. Each firm \( i \), when deciding about the quantity to produce and facing unknown demand, worries about two possibilities. It could be that the demand is actually very high, so the firm is losing profit by producing too little. The greatest such loss occurs when the demand is the highest, so \( P = \bar{P} \). Alternatively, it could be that the demand is actually very low, so the firm is losing profit by producing too much. The greatest such loss occurs when the demand is the lowest, so \( P = \bar{P} \). The firm thus chooses the best compromise \( q^*_i \) that balances these two losses, assuming that the other firm follows its equilibrium strategy \( q^*_{-i} \).

Remark 2. It is generally intractable to find a PBE in this game with such a rich set of possible demand functions. It can only be done under very specific, degenerate priors about the demand. For example, PBE can be found if the prior has support only on linear demand functions, \( P(q) = a - bq \). Note that when the slope is known, so \( b = \bar{b} = \bar{b} \), and the intercept \( a \) is uniformly distributed on \([\bar{a}, \bar{a}]\), then the firms’ strategies in PBE and PCE are identical and given by \( q^*_1 = q^*_2 = (\bar{a} + \bar{a})/(6\bar{b}) \).

Remark 3. Our equilibrium analysis can shed light on how the firms’ behavior changes in response to increasing uncertainty. For comparative statics, let us consider as a benchmark a linear demand function \( P_0(q) = a_0 - b_0q \). We normalize the constants \( a_0 \) and \( b_0 \) such that the monopoly profit is equal to 1, that is,
\[ \sup_{q \geq 0} (a_0 - b_0q)q = \frac{a_0^2}{4b_0} = 1. \]

Suppose that there is a small uncertainty of the magnitude \( \varepsilon \) about the demand relative to the benchmark. Specifically, for \( \varepsilon > 0 \) let \( P(q) \) satisfy (2) where
\[ P(q) = \left(1 - \frac{\varepsilon}{2}\right)a_0 - \left(1 + \frac{\varepsilon}{2}\right)b_0q \quad \text{and} \quad \bar{P}(q) = \left(1 + \frac{\varepsilon}{2}\right)a_0 - \left(1 - \frac{\varepsilon}{2}\right)b_0q. \]

Denote by \( q^\varepsilon = (q^*_1, q^*_2) \) the strategies of the PCE as given by Proposition 1. We then obtain
\[ \frac{dq^*_i}{d\varepsilon} = \frac{2\varepsilon}{3a_0} + O(\varepsilon^3) > 0. \]

So the firms optimally respond to the growing uncertainty about the demand by increasing their output, and do so at an increasing rate as \( \varepsilon \) grows. Next, consider the associated maximum losses
\[ l_i(q^*_i, q^*_{-i}) = \varepsilon^2 + O(\varepsilon^4), \quad i = 1, 2. \]

Moreover, if \( \varepsilon = 0.1 \), then \( l_i(q^*_i, q^*_{-i}) \approx 0.01 \). So the firms lose no more than about 1% of the maximum profit due to not knowing the demand.
3.2. Bertrand Duopoly with Private Costs. We now consider how two firms compete in prices when the cost of the rival firm is unknown.

There are two firms that produce a homogeneous good. Each firm \( i = 1, 2 \) chooses a price \( p_i \). Choices are made simultaneously. The consumers only buy from the firm that offers a lower price. So the quantity that firm \( i \) sells is given by

\[
q_i(p_i, p_{-i}) = \begin{cases} 
  Q(p_i), & \text{if } p_i < p_{-i}, \\
  Q(p_i)/2, & \text{if } p_i = p_{-i}, \\
  0, & \text{if } p_i > p_{-i},
\end{cases}
\]

where \( Q(p) \) is the demand function. For clarity of exposition we assume that the demand function is given by

\[
Q(p) = \max\left\{ a - \frac{p}{b}, 0 \right\}
\]

The cost producing \( q_i \) units is \( c_i q_i \). Each firm \( i \)'s profit is given by

\[
u_i(p_i, p_{-i}; c_i) = (p_i - c_i)q_i(p_i, p_{-i}), \quad i = 1, 2.
\]

Each firm \( i \) knows her own marginal cost \( c_i \) but not that of the other, and it is common knowledge that \( c_1, c_2 \in [0, \bar{c}] \), where \( 0 \leq \bar{c} \leq \bar{c} \leq a/2 \).

A firm \( i \)'s pricing strategy \( p_i^*(c_i) \) describes its choice of the price given its marginal cost \( c_i \).

For each marginal cost \( c_i \), firm \( i \)'s maximum loss of choosing a price \( p_i \) when facing pricing strategy \( p_{-i}^* \) of the other firm is given by

\[
l_i(p_i, p_{-i}^*; c_i) = \sup_{c_{-i} \in [c, \bar{c}]} \left( \sup_{p_{-i} \geq 0} u_i(p_i, p_{-i}^*(c_{-i}); c_i) - u_i(p_i, p_{-i}^*(c_{-i}); c_i) \right).
\]

The maximum loss describes how much more profit \( i \) could have obtained if it had known the other firm’s marginal cost \( c_{-i} \), anticipating the other firm to follow the pricing strategy \( p_{-i}^* \). Firm \( i \)'s best compromise given \( c_i \) is a price \( p_i^*(c_i) \) that achieves the lowest maximum loss for a given strategy \( p_{-i}^* \) of the other firm:

\[
p_i^*(c_i) \in \arg \min_{p_i \geq 0} l_i(p_i, p_{-i}^*; c_i).
\]

A strategy profile \((p_1^*, p_2^*)\) is a perfect compromise equilibrium if each firm \( i \) chooses a best compromise given its marginal cost \( c_i \) and the strategy \( p_{-i}^* \) of the other firm.

This application can be embedded in our formal setting as described in Section 2. As the firms’ marginal costs are not common knowledge, the pair \((c_1, c_2)\) is identified as the state. So the set of states is \( \mathcal{C} = [c, \bar{c}]^2 \). In the formal game, first nature chooses a state \((c_1, c_2)\), then each firm \( i \) observes its own cost \( c_i \), and then the two firms simultaneously choose their prices. A firm’s conceivable set contains all cost pairs that include their own cost. The posteriors are trivial, as this is a simultaneous move game.
Proposition 2. There exists a unique perfect compromise equilibrium whose strategy profile $p^* = (p_1^*, p_2^*)$ is given by

$$p_i^*(c_i) = \frac{1}{2} \left(a + c_i - \sqrt{(a - \bar{c})^2 + (\bar{c} - c_i)^2}\right), \quad i = 1, 2. \quad (5)$$

The associated maximum losses are

$$l_i(p_i^*(c_i), p_{-i}^*, c_i) = \frac{(a - \bar{c})(\bar{c} - c_i)}{2} \leq \frac{(a - \bar{c})(\bar{c} - \bar{c})}{2}, \quad i = 1, 2. \quad (6)$$

The proof is in Appendix A.3.

Let us discuss the strategic concerns underlying the PCE in this game. Each firm $i$, when deciding about the price $p_i > c_i$ and facing unknown cost of the other firm, worries about two possibilities. It could be that the other firm chooses a weakly lower price $p_{-i} \leq p_i$. Thus, firm $i$ could have obtained more profit by undercutting $p_{-i}$. The greatest such loss occurs when the other firm’s price marginally undercuts $p_i$. Alternatively, it could be that the other firm chooses a higher price, $p_{-i} > p_i$. Thus, firm $i$ is losing profit by charging too little. The greatest such loss occurs when the other firm’s cost is the highest possible, $\bar{c}$. The firm thus chooses the best compromise $p_i^*(c_i)$ that balances these two losses, assuming that the other firm follows its equilibrium strategy.

We find that the PCE price $p_i^*(c_i)$ is strictly increasing in $c_i$ and lies strictly above its marginal cost $c_i$ whenever $c_i < \bar{c}$. Moreover, $p_i^*(\bar{c}) = \bar{c}$. So, any sale with the cost below $\bar{c}$ leads to a positive profit. The fact that the price does not lie above $\bar{c}$ is intuitive. It is common knowledge that the costs are at most $\bar{c}$, so if the prices were above $\bar{c}$, each firm would have incentive to undercut the other, regardless of what its cost is. Also, the largest price cannot lie below $\bar{c}$, as a firm with cost $\bar{c}$ will charge the price $\bar{c}$ in order to ensure a loss equal to zero.

Note that the lowest price $p_i^*(\bar{c})$ is strictly positive, even if $\bar{c} = 0$. This is because when the price is very low, then the potential loss due to not undercutting the other firm is small, while the potential loss due to not setting a price much higher is large. This has an upward effect on prices.

Remark 4. It is generally intractable to find a PBE in this application under any reasonable prior, even in this simplest setting with linear demand and constant marginal costs. The PBE strategy profile for this simplest setting is implicitly defined by a differential equation with no closed form solution (see Spulber, 1995).

Remark 5. As in Section 3.1, our equilibrium analysis can shed light on how the firms’ behavior changes in response to increasing uncertainty. For comparative statics, let us consider as a benchmark marginal cost $c_0 = a/4$ (recall that we require $0 \leq c_i \leq a/2$, so $c_0 = a/4$ is the midpoint). We normalize the constants $a$ and $b$ of the demand function $Q(p) = (a - p)/b$ such that $a = 1$ and the monopoly profit is equal to 1, that is,

$$\sup_{p \geq 0} (p - c_0) \frac{a - p}{b} = \frac{(a - c_0)^2}{4b} = 1.$$
Suppose that there is a small uncertainty of the magnitude \( \varepsilon \) about the private cost relative to the benchmark. Specifically, for \( 0 < \varepsilon < 1 \) let \( c_i \in [\bar{c}, \bar{c}] \), \( i = 1, 2 \), where

\[
\bar{c} = \left(1 - \frac{\varepsilon}{2}\right)c_0 \quad \text{and} \quad \bar{c} = \left(1 + \frac{\varepsilon}{2}\right)c_0.
\]

Denote by \( p^\varepsilon = (p^\varepsilon_1, p^\varepsilon_2) \) the PCE strategy profile as given by Proposition 2. We then obtain

\[
\frac{dp^\varepsilon_i(c_i)}{d\varepsilon} = \frac{(a + c_i - 2\bar{c})c_0}{4\sqrt{(a - \bar{c})^2 + (\bar{c} - c_i)^2}} > 0,
\]

because, using our assumptions on the parameters,

\[
a + c_i - 2\bar{c} \geq a - 2\bar{c} = 1 - 2 \left(1 + \frac{\varepsilon}{2}\right)c_0 = \frac{1}{4}(2 - \varepsilon) > 0.
\]

So the firms optimally respond to the growing uncertainty about the demand by increasing their prices. They become less competitive. Next, consider the associated maximum losses:

\[
l_i(p^\varepsilon_i(c_i), p^\varepsilon_{-i}, c_i) \leq \frac{3\varepsilon}{32} - \frac{\varepsilon^2}{64}, \quad i = 1, 2.
\]

Moreover, if \( \varepsilon = 0.1 \), then the maximum losses are bounded by 0.01. So the firms lose no more than about 1% of the maximum profit due to not knowing the cost of the other firm.

### 3.3. Job Market Signaling

Here we investigate Spence’s job market signaling when the worker’s cost of education is unknown to the firms.

There is a single worker and two firms. The worker has productivity \( \theta \in [0, 1] \). The worker publicly chooses a level of education, either low \( (e_L) \) or high \( (e_H) \), to signal her productivity to the firms. The cost of low education is zero. The cost of high education depends on the worker’s productivity and is given by \( c(\theta) \). The firms observe the worker’s education level \( e \) and simultaneously offer wages \( w_1 \) and \( w_2 \). The worker chooses the better of the two wages. Her payoff is given by

\[
v(w_1, w_2, e; \theta, c) = \max\{w_1, w_2\} - \begin{cases} 0, & \text{if } e = e_L, \\ c(\theta), & \text{if } e = e_H. \end{cases}
\]

Each firm \( i \)’s payoff is given by

\[
u_i(w_1, w_{-i}; \theta) = \begin{cases} \theta - w_i, & \text{if } w_i > w_{-i}, \\ (\theta - w_i)/2, & \text{if } w_i = w_{-i}, \\ 0, & \text{if } w_i < w_{-i}. \end{cases}
\]

The worker knows her productivity type \( \theta \) and the cost of high education \( c(\theta) \). The firms know neither. They only know that \( \theta \in [0, 1] \) and that the cost function \( c \) is bounded by two functions, \( c \) and \( \bar{c} \). Specifically, let

\[
c(\theta) = 1 - b\theta \quad \text{and} \quad \bar{c}(\theta) = 1 + \delta - b\theta, \quad \text{where } 0 \leq \delta < b < 1.
\]
The firms know that
\[ c(\theta) \] is strictly decreasing,
\[ c(\theta) \leq c(\theta) \leq \bar{c}(\theta) \quad \text{for all } \theta \in [0, 1]. \]

Define \( \Omega \) to be the set of all pairs \((\theta, c)\) of productivities \(\theta \in [0, 1]\) and cost functions \(c(\theta)\) that satisfy (7).

The worker’s strategy \(e^*(\theta, c)\) describes her choice of the education level for each pair \((\theta, c) \in \Omega\). Each firm \(i\)’s strategy \(w_i^*(e)\) describes its wage offer conditional on each education level \(e \in \{e_L, e_H\}\). In addition, each firm has a conceivable set \(B_i(e)\). This is the set of all pairs \((\theta, c)\) that firm \(i\) considers possible after observing the level of education \(e \in \{e_L, e_H\}\).

A conceivable set \(B_i(e)\) is consistent with the worker’s strategy \(e^*\) if it includes all pairs \((\theta, c)\) under which the worker chooses \(e\), so \((\theta, c) \in B_i(e)\) if \(e^*(\theta, c) = e\).

For each education level \(e\), firm \(i\)’s maximum loss of choosing wage \(w_i\) when the other firm chooses the wage according to strategy its \(w_i^*\) is given by
\[
 l_i(w_i, w_i^*; e) = \sup_{(\theta,c) \in B_i(e)} \left( \sup_{w_i' \geq 0} u_i(w_i', w_i^*(e); \theta) - u_i(w_i, w_i^*(e); \theta) \right).
\]

The maximum loss describes how much more profit firm \(i\) could have obtained if it had known the true productivity of the worker, anticipating that the other firm follows its strategy \(w_i^*\). Firm \(i\)’s best compromise given \(e\) is a wage \(w_i^*(e)\) that achieves the lowest maximum loss for a given strategy \(w_i^*\) of the other firm:
\[
 w_i^*(e) \in \arg \min_{w_i \geq 0} l_i(w_i, w_i^*; e). \tag{8}
\]

Observe that the worker has complete information. There is no need for a compromise. So, the worker simply makes a best-response choice:
\[
 e_i^*(\theta, c) \in \arg \max_{e \in \{e_L, e_H\}} v(w_1^*(e), w_2^*(e), e; \theta, c). \tag{9}
\]

A profile \((e^*, w_1^*, w_2^*, B_1, B_2)\) of strategies and conceivable sets is a perfect compromise equilibrium (PCE) if two conditions hold. First, the strategies satisfy (8) and (9), so each firm \(i\) chooses a best compromise, and the worker chooses a best response to the strategies of the others. Second, the firms’ conceivable sets are consistent with the worker’s strategy \(e^*\).

This application can be embedded in our formal setting as described in Section 2. As the pair \((\theta, c)\) is unknown to the firms, it is identified as the state. So the set of states is \(\Omega\). In the formal game, first nature chooses a state \((\theta, c) \in \Omega\). Then the worker observes \((\theta, c)\) and chooses an education level \(e\). Finally, the firms observe the worker’s choice \(e\) and simultaneously choose their wages. Conceivable sets for each firm \(i\) are given by \(B_i\). Posteriors are trivial, because the worker plays a pure strategy, and there are no chance moves.

A PCE is pooling if the worker chooses the same level of education for all \((\theta, c) \in \Omega\). A PCE is separating if the set \(\Omega\) can be partitioned into two subsets such that a different level of education is chosen in each element of the partition.
Proposition 3. (i) There exists a pooling PCE in which the worker chooses low education, so
\[ e^*(\theta) = e_L \text{ for all } (\theta, c) \in \Omega, \]
and the firms’ wages are given by
\[ w^*_i(e_H) = w^*_i(e_L) = \frac{1}{2}, \quad i = 1, 2. \]
After each observed education level \( e \), each firm believes about the worker’s productivity that \( \theta \in [0, 1] \).

(ii) If \( \delta \geq 2b^2 - b \), then a separating PCE does not exist.

(iii) If \( \delta < 2b^2 - b \), then there exists a separating PCE in which the worker chooses high education if and only if her cost is at most \( \frac{1}{2b}(b - \delta) \), so for all \( (\theta, c) \in \Omega \)
\[ e^*(\theta, c) = \begin{cases} e_H, & \text{if } c(\theta) \leq \frac{1}{2b}(b - \delta), \\ e_L, & \text{if } c(\theta) > \frac{1}{2b}(b - \delta), \end{cases} \]
and the firms’ wages are given by
\[ w^*_i(e_H) = \frac{1}{2} + \frac{b + \delta}{4b^2} \quad \text{and} \quad w^*_i(e_L) = \frac{\delta}{2b} + \frac{b + \delta}{4b^2}, \quad i = 1, 2. \] (10)
After each observed education level \( e \), each firm believes about the worker’s productivity that
\[ \theta \in \left[ 0, \frac{b + \delta}{2b^2} + \frac{\delta}{b} \right] \text{ if } e = e_L, \quad \text{and} \quad \theta \in \left[ \frac{b + \delta}{2b^2}, 1 \right] \text{ if } e = e_H. \] (11)

The proof is in Appendix A.4.

Let us discuss the strategic concerns underlying these two PCE. In either PCE, each firm \( i \), when deciding about the wage offer \( w_i \) and facing unknown productivity of the worker, worries about two possibilities. It could be that the productivity is high, so offering a wage that is marginally greater than that of the competitor would improve profit. The greatest such loss occurs when the productivity is the highest possible. Alternatively, it could be that the productivity is low, so offering a wage that is smaller than the competitor’s would cut the loss. The greatest such loss occurs when the productivity is the lowest possible. The firm thus offers the best compromise wage that balances these two losses, assuming that the other firm follows its equilibrium strategy.

In equilibrium, the firms offer the same wage and do not try to outbid each other, because they can lose equal amounts by stealing the worker for themselves if she is unproductive and by giving up the worker if she is highly productive.

An essential detail in the above considerations is that the greatest and smallest productivities are now endogenous and can depend on the level of education \( e \) that the worker chooses. In the pooling equilibrium, \( e = e_L \) does not provide any useful information, so all productivity types are possible. However, in the separating equilibrium, the firms believe that the productivity belongs to a different interval when observing a different level of education. For example, if \( b = 1 \) and \( \delta = 1/4 \),
then the firms believe that $\theta \leq 7/8$ if the education is low, and that $\theta \geq 5/8$ if the education is high.

Observe that, among the workers with productivity $\theta \in [5/8, 7/8]$, some choose low education, while others choose high education. This overlap is due to the richness of the state space. The same productivity type $\theta$ can have different costs of education $c(\theta)$ that can fall below or above the threshold at which high education is profitable. Clearly, this result cannot emerge in the traditional setting where there are only two types of workers.

The parameter $\delta$ captures the firms’ uncertainty about the worker’s cost of education conditional on knowing her productivity type. As $\delta$ goes up, the range of productivity types that cannot be identified with a specific cost of education increases. When $\delta$ is sufficiently large, education signaling is not very informative. A costly signal does not allow to rule out low productivity types, and the separating PCE does not exist.

3.4. Bilateral Trade with Common Value. We now examine bilateral trade with common value and private information. In this example we illustrate the role of the order of moves when traders are asymmetrically informed.\footnote{The traditional bilateral trade model with private values and simultaneous offers is analyzed in Appendix B.1.}

A seller wants to sell an indivisible good to a buyer. The value $v$ of the good is the same for each of them. If the good is traded at some price $p$, then the buyer obtains $v - p$ and the seller obtains $p - v$. If the good is not traded, then both parties obtain zero.

The value $v$ is given by

$$v = \frac{x + y}{2}, \quad x \in [0, 1] \text{ and } y \in [0, 1],$$

so $v \in [0, 1]$. Only the seller observes $x$, while neither trader observes $y$. Thus $x$ represents the seller’s private information and $y$ represents the common uncertainty among the two traders. Both traders know that $x \in [0, 1]$ and $y \in [0, 1]$.

We consider the take-it-or-leave-it protocol, in which the proposer can be either buyer or seller. The protocol is as follows. First, the seller observes her private information $x$. Then one trader (proposer) offers a price $p \in [0, 1]$. Finally, the other trader (responder) observes the price offer and chooses a probability $\alpha \in [0, 1]$ with which the offer is accepted, in which case the trade takes place. If the offer is not accepted, then the trade does not take place.

This application can be embedded in our formal setting as described in Section 2. We identify the state with $(x, y)$. So the set of states is $\Omega = [0, 1]^2$. In the formal game, first nature chooses a state $(x, y)$. Then the traders proceed as described in the bargaining protocol. Each trader’s conceivable set contains all pairs of $(x, y)$ that are possible given the trader’s information. The posteriors are trivial, because the proposer plays a pure strategy, and there are no chance moves.
3.4.1. Proposer is Buyer (Uninformed Trader). We first assume that the buyer is the proposer and the seller is the responder. Because the buyer is uninformed, her strategy is a choice of a price $p \in [0, 1]$. The seller’s strategy is an acceptance probability $\alpha^*(x, p)$ that depends on her private information $x$ and the offered price $p$. Note that the offer of the buyer does not contain any information about the value, so there is no signaling.

The seller’s maximum loss of an accepting an offer $p$ with probability $\alpha$, given private information $x$, is

$$l_s(\alpha; x, p) = \sup_{y \in [0, 1]} \left( \max \left\{ p - \frac{x + y}{2}, 0 \right\} - \left( p - \frac{x + y}{2} \right) \alpha \right).$$

It describes how much more the seller could have obtained if she knew the missing information $y$. The buyer’s maximum loss of offering price $p$, given the seller’s acceptance strategy $\alpha^*$, is

$$l_b(p, \alpha^*) = \sup_{(x, y) \in [0, 1]^2} \left( \sup_{p' \in [0, 1]} \left( \frac{x + y}{2} - p' \right) \alpha^*(x, p') - \left( \frac{x + y}{2} - p \right) \alpha^*(x, p) \right).$$

It describes how much more the buyer could have obtained if she knew $x$ and $y$, anticipating that the seller would follow her strategy $\alpha^*$. Each trader’s best compromise is a choice that achieves the lowest maximum loss for a given strategy of the other trader. A strategy profile $(p^*, \alpha^*)$ is a perfect compromise equilibrium if each trader chooses a best compromise given the strategy of the other trader.

Consider first how the seller finds an acceptance probability $\alpha$ as a best compromise to a price offer in an abstract setting where the seller believes that the value is in an interval $[v_0, v_1]$.

**Lemma 1.** Let price be $p \in [0, 1]$. Suppose that the seller believes that $v \in [v_0, v_1]$. Then the seller’s best compromise acceptance probability is

$$\alpha = \begin{cases} 
0, & \text{if } p \leq v_0, \\
(p - v_0)/(v_1 - v_0), & \text{if } v_0 < p < v_1, \\
1, & \text{if } p \geq v_1.
\end{cases}$$

The seller’s associated maximum loss is $\alpha \max\{v_1 - p, 0\}$.

The proof is in Appendix A.5.

The intuition for this result is simple. Of course, if $p \geq v_1$, then the payoff $p - v$ is nonnegative for all $v \in [v_0, v_1]$, so the optimal choice is to accept the offer. Similarly, if $p \leq v_0$, then the optimal choice is to reject the offer. However, if $v_0 < p < v_1$, then, for a given choice of the acceptance probability $\alpha$, the seller worries about two possibilities. First, the value could be smaller than the price, so $p - v > 0$. In that case, she could have obtained a better payoff by accepting the proposal with certainty. The greatest such loss occurs when $v = v_0$, and it is equal to

$$(p - v_0) - \alpha(p - v_0) = (1 - \alpha)(p - v_0).$$
Alternatively, the value could be greater than the price, so \( p - v < 0 \). In that case, she could have obtained a better payoff by rejecting the proposal with certainty, and thus getting zero. The greatest such loss occurs when \( v = v_1 \), and it is equal to

\[
0 - \alpha(p - v_1) = \alpha(v_1 - p).
\]

The seller chooses \( \alpha \) as a best compromise to balances these two losses, that is, \( \alpha \) solves the equation \((1 - \alpha)(p - v_0) = \alpha(v_1 - p)\).

We now present a PCE of this game.

**Proposition 4.** There exists a perfect compromise equilibrium, in which the buyer offers \( p^* = 1/4 \), and the seller accepts the offer with probability

\[
\alpha^*(x, p) = \begin{cases} 
0, & \text{if } x \geq 2p, \\
2p - x, & \text{if } 2p - 1 < x < 2p, \\
1, & \text{if } x \leq 2p - 1.
\end{cases}
\] (12)

The proof is in Appendix A.6.

Under this PCE, trade can occurs with positive probability. The seller accepts the equilibrium offer of 1/4 with probability \( \max\{1/2 - x, 0\} \).

Let us discuss how this PCE is computed. In the second stage, the seller, who has observed \( x \) and \( p \) and has no information about \( y \in [0, 1] \), believes that \( v = (x + y)/2 \) belongs to the interval \([x/2, (1 + x)/2]\). Therefore, her acceptance strategy \( \alpha^* \) is given by Lemma 1 with \([v_0, v_1] = [x/2, (1 + x)/2]\).

In the first stage, the buyer considers different possible realizations of the common uncertainty \( y \) and of the seller’s private information \( x \). As the price goes up, on the one hand the buyer’s gain from trade decreases, but on the other hand, the probability that the seller accepts this price increases. Unlike in our earlier examples, here the computation of the maximum loss involves more than just checking the extreme cases. However, notice that the price should not be too low, as this offer is likely to be rejected, so the buyer has a high loss when the common value of the good is high. The price should not be too high either, as this offer is likely to be accepted, so the buyer has a high loss when the common value of the good is low. As best compromise, the buyer chooses the price that balances these two considerations, anticipating the seller’s equilibrium behavior in the second stage.

3.4.2. Proposer is Seller (Informed Trader). Now we assume that the seller is the proposer and the buyer is the responder. Because the seller observes \( x \) and \( p \) and has information about \( y \), her pricing strategy \( p^*(x) \) depends upon \( x \). The buyer then responds by an acceptance probability \( \alpha^*(p) \) that depends on the offered price \( p \).

Unlike when the buyer was the proposer, here the seller’s price can be informative about the seller’s private information \( x \). Let \( B(p) \) be the buyer’s conceivable set that describes what pairs of \((x, y)\) the buyer considers possible after observing the seller’s move. A conceivable set \( B(p) \) is consistent with the seller’s strategy \( p^* \) if it includes all pairs \((x, y)\) under which the seller chooses \( p \), so \((x, y) \in B(p)\) if \( p^*(x) = p \).
The buyer’s maximum loss from accepting $p$ with probability $\alpha$ is

$$l_b(\alpha; p) = \sup_{(x,y) \in B(p)} \left( \max \left\{ \frac{x + y}{2} - p, 0 \right\} - \left( \frac{x + y}{2} - p \right) \alpha \right).$$

The seller’s maximum loss of offering price $p$, given her private information $x$ and the buyer’s acceptance strategy $\alpha^*$, is

$$l_s(p, \alpha^*; x) = \sup_{y \in [0,1]} \left( \sup_{p' \in [0,1]} \left( p' - \frac{x + y}{2} \right) \alpha^*(p') - \left( p - \frac{x + y}{2} \right) \alpha^*(p) \right).$$

Each trader’s best compromise is a choice that achieves the lowest maximum loss for a given strategy of the trader. A strategy profile $(p^*, \alpha^*)$ is a perfect compromise equilibrium if each trader chooses a best compromise given the strategy of the other trader, and the buyer’s conceivable sets are consistent.

We now present a PCE. In the proposition below, instead of expressing what pairs $(x, y)$ the buyer includes in the conceivable set, we simply state what values of $v = (x + y)/2$ the buyer considers as conceivable.

**Proposition 5.** There exists a perfect compromise equilibrium in which the seller’s offer is

$$p^*(x) = \frac{3}{4} \text{ for all } x \in [0,1],$$

the buyer believes that $v \in [0,1]$ after observing $p = 3/4$ and that $v \in [0,1/2]$ after observing $p \neq 3/4$, and her acceptance probability is

$$\alpha^*(p) = \begin{cases} 1/4, & \text{if } p = 3/4, \\ \max\{1 - 2p, 0\}, & \text{if } p \neq 3/4. \end{cases}$$

The proof is in Appendix A.7.

This PCE is pooling, in the sense that the equilibrium behavior is independent of the seller’s private information. In the second stage, the buyer, who has observed $p$, knows that in equilibrium the seller must choose $p = 3/4$. This price reveals no information about $x$. So the buyer cannot rule out any values, thus believing that $v \in [0,1]$. The buyer’s acceptance strategy is derived analogously to the seller’s in Lemma 1. So, the buyer accepts the offer with probability 1/4. Alternatively, if the buyer observes $p \neq 3/4$, which cannot happen in equilibrium, then there are no constraints on what values are conceivable. In this case the equilibrium conceivable set is specified to be such that $x = 0$ and $y \in [0,1]$. The buyer thus believes that $v \in [0,1/2]$ and accepts the offer with probability $\max\{1 - 2p, 0\}$.

In the first stage, the seller observes $x$, but still faces different possible realizations of the common uncertainty $y$. She anticipates the buyer’s acceptance of the price of $p = 3/4$ with probability 1/4, and the acceptance of a price $p \neq 3/4$ with probability $\max\{1 - 2p, 0\}$. The price of 3/4 has the property that the maximum loss of choosing $p = 3/4$ is never greater than the maximum loss of choosing any other price, regardless of the seller’s private value $x$.

3.4.3. **Discussion.** Propositions 4 and 5 stand in stark contrast to the no-trade theorem under common values as predicted by PBE (Milgrom and Stokey, 1982).
We observe that trade can happen with a positive probability in a PCE. This is true when the (uninformed) buyer is the proposer as well as when the (informed) seller is the proposer. When the buyer is the proposer, the probability of trade is

\[ \alpha^*(x, p^*) = \max\{1/2 - x, 0\} > 0 \]

for all \( x < 1/2 \). When the seller is the proposer, the probability of trade is \( 1/4 \), regardless of the seller’s private information. The trade is possible because the traders cannot rule out the possibility of two opposing contingencies: winning and losing from trade. They do not want to miss a winning opportunity, but also they do not want to lose from trade. They compromise by trading with a positive probability when the informed trader moves first, as well as in many cases when the uninformed trader moves first.

4. Conclusion

We introduce the concept of perfect compromise equilibrium (PCE) as an alternative to perfect Bayesian equilibrium (PBE) for solving sequential games when there is uncertainty about some of the primitives. These primitives can be specific characteristics of other players or the environment in which the game takes place. Traditionally, following Harsanyi (1967), such uncertainty is reduced to risk. A state is introduced to capture the details underlying this uncertainty. At the outset of the game a move of nature determines this state. This move is drawn from a distribution that is commonly known among the players in the game. We follow the same approach except for the last assumption, and assume instead that the distribution determining the state is unknown.

With our concept, dynamic strategic decision making can be illustrated in the traditional fashion with a game tree. Beliefs are added whenever an information set contains more than one node. This allows to evaluate choices at information sets that cannot be reached. These beliefs are conditional on a state to separate what can happen in the different states. Bayes’ rule is still in place, applying conditional on a state. A set of conceivable states is added to each information set in order to track any resolution of uncertainty. The concept of best compromise determines how to evaluate a strategy when looking at its performance across different conceivable states. This stands in contrast to the classic PBE setting where strategies are evaluated using a prior over the different states.

The description of the state governs the type of uncertainty that is modeled. States can describe values of market participants, thereby allowing for many different possible environments. They can specify distributions of such values in some small neighborhood, thereby modeling slight uncertainty around a given understanding of the world. States can also include features of the choices of a player, thereby incorporating strategic uncertainty into the analysis. In particular, our assumption of strategic certainty comes without loss of generality. Any uncertainty can be captured in the definition of a state. The common strategy profile describes all details that can be predicted if the state were known, provided this strategy profile is common knowledge.
Uncertainty seems to mean that details are hard to describe. And yet traditional models focus on two types of workers, high and low, or assume linear demand functions. Uncertainty seems to preclude that players agree on likelihoods of uncertain events and yet this is done in PBE. PCE opens the door to understanding more realistic uncertainty.

We demonstrate the usefulness of our solution concept in relevant economic examples. The underlying game trees are simple while the uncertainty is rich. This richness, such as allowing for any demand function in a neighborhood, precludes a tractable analysis of PBE. PCE yields tractable results with simple proofs as players focus on extreme situations, allowing them to ignore intermediate constellations. New insights come to light.

The traditional PBE framework reveals a different solution for each prior. Such flexibility can be useful to fit data. But flexibility in terms of a multitude of different answers gives little guidance to those who need to make choices. One easily looses the big picture if there are many details that determine what happens. On a more abstract level, traditional PBE analysis reveals parsimonious results by limiting attention to only few types of players. In contrast, PCE generates parsimonious results by forcing players to find a compromise in many different situations.

Acceptance of the common knowledge assumption is dwindling. The literature on decision making and game playing under uncertainty has now developed alternative concepts. We hope to add to this literature. Numerous paths to future research open up in a search for new insights and for a clearer exposition of existing understanding of economic and strategic principles.

Appendix A. Proofs.

A.1. Proof of Theorem 1. Suppose that the strategy $s_0$ of the nonstrategic player 0 is fully mixed. This is without loss of generality, as any zero-probability action of player 0 can be removed from the game tree.

We now introduce the notion of feasibility. A state $\omega$ is feasible at information set $\phi \in \Phi$ if there exists a path in the game tree $G$ from the initial node $\phi_0$ to $\phi$ in which nature chooses $\omega$ at $\phi_0$. Let $\bar{B}(\phi)$ be the set of all feasible states at $\phi$.

Let $S_{s_0}$ be the set of strategy profiles $s$ where the strategy of the nonstrategic player 0 is exogenously given by $s_0$. Let conceivable sets be equal to feasible sets, $B(\phi) = \bar{B}(\phi)$ for all $\phi \in \Phi$. This profile of conceivable sets is consistent with every strategy profile $s \in S_{s_0}$ (see Definition 1). Let $B$ be the set of belief systems $\beta = (\beta_\omega)_{\omega \in \Omega}$.

We now argue that there exists a PCE $(s, \bar{B}, \beta)$ where $s \in S_{s_0}$ and $\beta \in B$. Note that we fix the profile of conceivable sets to be equal to $\bar{B}$.

Consider an arbitrary $(s, \bar{B}, \beta)$ that satisfies $s \in S_{s_0}$ and $\beta \in B$. Let $(s'_\phi, s_{-\phi})$ denote the strategy profile where $s'_\phi \in \Delta(A(\phi))$ is played at information set $\phi$ and $s_{-\phi}$ is the profile of strategies at all information sets other than $\phi$. For each
information set \( \phi \in \Phi \) let \( U_\phi(s; \beta) \) be the negative of the maximum loss at \( \phi \), so

\[
U_\phi(s; \beta) = -\sup_{\omega \in \overline{B}(\phi)} \left( \sup_{a_\phi \in \Delta(A(\phi))} \bar{u}_i(a_\phi, s_{-\phi}; \phi, \omega, \beta_\omega) - \bar{u}_i(s_\phi, s_{-\phi}; \phi, \omega, \beta_\omega) \right).
\]

We now construct an augmented game \( (\Phi, G, U, s_0, \mu) \) as follows. Let each information set \( \phi \in \Phi \) be associated with a different player, so the set of players is the set of information sets \( \Phi \). The game tree \( G \) remains unchanged. Nature moves first by choosing a state \( \omega \in \Omega = A(\phi_0) \) in the initial node \( \phi_0 \). We now model the choice of nature by a given distribution \( \mu \) over the states. We assume that \( \mu \) has full support over nature’s actions in \( \Omega \). Each player \( \phi \in \Phi \) moves only once, at her information set \( \phi \), by choosing an action from the set \( A(\phi) \).

A strategy profile \( s \) describes a choice \( s_\phi \in \Delta(A(\phi)) \) of each player \( \phi \). For each information set \( \phi \) that belongs to the nonstrategic player 0 in the original game, the strategy at \( \phi \) is exogenously given by \( s_0(\phi) \). A posterior \( \beta_\omega(\phi) \) is the probability distribution over decision nodes in \( \phi \) conditional on state \( \omega \). The interim payoff of each player \( \phi \in \Phi \) at the information set \( \phi \) is given by \( U_\phi(s; \beta) \), and \( U = (U_\phi)_{\phi \in \Phi} \).

The augmented game \( (\Phi, G, U, s_0, \mu) \) can be seen as a game of incomplete information with a nonstandard specification of the players’ payoffs. While in a standard game the payoffs are ex post and specified at each terminal node, in this augmented game the payoff \( U_\phi \) of each player \( \phi \in \Phi \) is interim and specified at the information set where the player makes a move. Because each player moves only once, the specification of the interim payoffs is sufficient to apply the concept of PBE or sequential equilibrium to the augmented game.

Another nonstandard feature of the augmented game is that each player’s interim payoff \( U_\phi(s; \beta) \) is independent of nature’s choice \( \omega \). That is, for each state \( \omega \in \overline{B}(\phi) \) the interim payoff \( U_\phi(s; \beta) \) at \( \phi \) is the same, and for each state \( \omega \notin \overline{B}(\phi) \) the information set \( \phi \) cannot be reached. So, nature’s distribution over states \( \mu \) does not affect the best-response actions by the players, it only affects the likelihood of reaching different information sets in the game tree.

Observe that maximizing \( U_\phi(s'_\phi, s_{-\phi}; \beta) \) with respect to player \( \phi \)’s own decision \( s'_\phi \in \Delta(A(\phi)) \) is the same as minimizing the maximum loss at \( \phi \) in the original game. Consequently, if \( (s^*, \beta^*) \) is a sequential equilibrium of the augmented game, then \( (s^*, B, \beta^*) \) is a PCE of the original game. The existence of PCE follows from the existence of sequential equilibrium for finite games. We refer the reader to Chakrabarti and Topolyan (2016) for the backward-induction proof of existence of sequential equilibrium that uses interim payoffs at information sets to determine players’ best-response correspondences.

□

A.2. Proof of Proposition 1. For derivations, we assume that the quantities and the price are always nonnegative, and then we verify that this is indeed the case in equilibrium.

Let \( x^*_i(q_{-i}, P) \) be a best response strategy of player \( i \) given the knowledge of \( q_{-i} \) and the inverse demand function \( P \). The loss of firm \( i \) from choosing quantity \( q_i \),
given \( q_{i}, \) and \( P, \) is denoted by \( \Delta u_{i}(q_{i}, q_{-i}; P) \) and given by

\[
\Delta u_{i}(q_{i}, q_{-i}; P) = P(x_{i}^{*}(q_{i}, P) + q_{i})x_{i}^{*}(q_{i}, P) - P(q_{i} + q_{i})q_{i}.
\]

By (2), the marginal revenue of firm \( i \) satisfies

\[
P'(q_{i} + q_{-i}) + P'(q_{i} + q_{-i})q_{i} \leq P'(q_{i} + q_{-i}) + P'(q_{i} + q_{-i})q_{i} \leq \bar{P}(q_{i} + q_{-i}) + \bar{P}'(q_{i} + q_{-i})q_{i}.
\]

Therefore, for given \( q_{j} \) and \( P, \) the best-response quantity \( x_{i}^{*}(q_{i}, P) \) of firm \( i \) always lies between \( x_{i}^{*}(q_{i}, P) \) and \( x_{i}^{*}(q_{i}, \bar{P}). \) While the profit function need not be concave in general, it is concave when \( P = \bar{P} \) or when \( P = \bar{P}. \) So the highest loss will always be attained in one of these two extreme cases:

\[
l_{i}(q_{i}, q_{-i}) = \sup_{P} \Delta u_{i}(q_{i}, q_{-i}; P) = \max\{\Delta u_{i}(q_{i}, q_{-i}; P), \Delta u_{i}(q_{i}, q_{-i}; \bar{P})\}.
\]

It is easy to see that the maximum loss is minimized by balancing the two expressions under the maximum:

\[
\Delta u_{i}(q_{i}, q_{-i}; \bar{P}) = \Delta u_{i}(q_{i}, q_{-i}; \bar{P}).
\]

Substituting \( P \) and \( \bar{P} \) and simplifying the expressions yields the equation

\[
\frac{(\bar{a} - \bar{b}q_{i})}{4b} - (\bar{a} - \bar{b}(q_{i} + q_{-i}))q_{i} = \frac{(a - bq_{i})}{4b} - (a - b(q_{i} + q_{-i}))q_{i}.
\]

Solving for \( q_{i} \) yields

\[
q_{i}^{*} = \frac{a\sqrt{b} + \bar{a}\sqrt{b}}{2(b\sqrt{b} + \bar{b}\sqrt{b})} - \frac{q_{j}}{2}, \quad i = 1, 2.
\]

Solving this pair of equations for \((q_{1}^{*}, q_{2}^{*})\), we find (3). It is easy to verify that under our assumptions, \( q_{i}^{*} > 0, \) and moreover, \( P(q_{1}^{*} + q_{2}^{*}) \geq \bar{P}(q_{1}^{*} + q_{2}^{*}) > 0. \) Substituting the solution into (13) yields the maximum loss of each firm (4).

### A.3. Proof of Proposition 2.

For derivations, we assume that each firm prices at or above marginal cost, and then we verify that this is indeed the case in equilibrium.

Consider firm \( i \) with type \( c_{i} \in [\bar{c}, \bar{c}]. \) Let \( p^{m}(c_{i}) \) be the monopoly price, so \( p^{m}(c_{i}) = (a + c_{i})/2. \) Since we have assumed that \( \bar{c} \leq a/2, \) this means that \( p^{m}(c_{i}) \geq \bar{c} \) for all \( c_{i}. \) The monopoly profit is \((a - c_{i})^{2}/(4b)\).

Fix the other firm’s strategy \( p_{-i}(c_{-i}) \) and let \( \bar{p} \) be the maximum price of the other firm, so \( \bar{p} = \sup_{c_{-i} \in [\bar{c}, \bar{c}]} p_{-i}(c_{-i}). \) Given the other firm’s cost \( c_{-i}, \) and thus the price \( p_{-i} = p_{-i}^{*}(c_{-i}), \) firm \( i \)‘s maximum profit is

\[
u_{i}^{*}(p_{-i}; c_{i}) = \sup_{x_{i} \geq 0} u_{i}(x_{i}, p_{-i}; c_{i}) = \begin{cases} 0, & \text{if } p_{-i} \leq c_{i}, \\ \frac{(a - p_{-i})}{b} - \frac{(a - c_{i})^{2}}{4b}, & \text{if } c_{i} < p_{-i} \leq p^{m}(c_{i}), \\ \frac{(a - c_{i})^{2}}{4b}, & \text{if } p_{-i} > p^{m}(c_{i}). \end{cases}
\]

\[
= \max \left\{ 0, \frac{(a - p_{-i})}{b} - \frac{(a - c_{i})^{2}}{4b} \right\}.
\]
Let $p_i$ be a price of firm $i$. We now find the maximum loss of firm $i$ from choosing $p_i$, given its marginal cost $c_i$ and the strategy $p^*_i$ of the other firm. There are three cases.

First, suppose that $p_{-i} \leq c_i \leq p_i$. Then firm $i$ cannot make positive profit, so $p_i$ is a best response. Thus, firm $i$ behaves optimally in this case, so the loss is zero.

Second, suppose that $c_i < p_{-i} \leq p_i$. Then firm $i$ could have been better off by marginally undercutting $p_{-i}$. Maximizing the loss over $p_{-i} \in (c_i, p_i]$, we obtain

$$
\sup_{p_{-i} \in (c_i, p_i]} (u_i^*(p_{-i}; c_i) - u_i(p_i, p_{-i}; c_i)) = \begin{cases}
(p_i - c_i) \frac{a - p_i}{b}, & \text{if } p_i \leq p^m(c_i), \\
\frac{(a - c_i)^2}{4b}, & \text{if } p_i > p^m(c_i).
\end{cases}
$$

(14)

Third, suppose that $p_i < p_{-i}$. Then firm $i$ could have made more profit by increasing its price, so its maximum loss is

$$
\sup_{p_{-i} \in [p_i, \bar{p}]} (u_i^*(p_{-i}; c_i) - u_i(p_i, p_{-i}; c_i)) = u_i^*(\bar{p}, c_i) - u_i(p_i, \bar{p}; c_i)
$$

\begin{equation}
= -(p_i - c_i) \frac{a - p_i}{b} + \begin{cases}
(p_i - c_i) \frac{a - \bar{p}}{b}, & \text{if } p_i \leq p^m(c_i), \\
\frac{(a - c_i)^2}{4b}, & \text{if } p_i > p^m(c_i).
\end{cases}
\end{equation}

(15)

To minimize the maximum loss, we need to minimize the greater of the expressions in (14) and (15). Observe that, by the definition of $p^m(c_i)$, the right-hand side in (14) is constant and the right-hand side in (15) is strictly increasing in $p_i$ for $p_i > p^m(c_i)$. So we only need to consider $p_i \leq p^m(c_i)$. Under this assumption, the greater of the expressions in (14) and (15) can be simplified to

$$l_i(p_i, p^*_i; c_i) = \max \left\{ (p_i - c_i) \frac{a - p_i}{b}, (\bar{p} - c_i) \frac{a - \bar{p}}{b} - (p_i - c_i) \frac{a - p_i}{b} \right\}. $$

Because one expression is increasing and the other is decreasing in $p_i$ for $p_i \leq p^m(c_i)$, the maximum loss is minimized at the solution of

$$
(p_i - c_i) \frac{a - p_i}{b} = (\bar{p} - c_i) \frac{a - \bar{p}}{b} - (p_i - c_i) \frac{a - p_i}{b}.
$$

(16)

Solving the above for $p^*_i(c_i)$, we obtain (5).

To see that $p^*_i(c_i) \geq c_i$, observe that

$$p^*_i(c_i) - c_i = \frac{1}{2} \left( a - c_i - \sqrt{(a - c)^2 + (\bar{c} - c_i)^2} \right) \geq 0
$$

by the triangle inequality and $a > \bar{c} \geq c_i$. Moreover, $p^*_i(c_i) > c_i$ when $c_i < \bar{c}$, and $p^*_i(\bar{c}) = \bar{c}$. Finally, substituting $p^*_i(c_i)$ into the maximum loss expression in (16) yields (6).

A.4. Proof of Proposition 3. First we find the equilibrium wages $w^H$ and $w^L$ after the worker’s level of education $e_H$ and $e_L$. For each $j = L, H$, the firms have a conceivable set $B(e_j) \subset \Omega$, that is, the set of states $(\theta, c) \in \Omega$ that the firms consider possible. Let $\theta_j$ and $\bar{\theta}_j$ be the lowest and highest productivity levels given $e_j$, so

$$
\theta_j = \inf \{ \theta : (\theta, c) \in B(e_j) \} \quad \text{and} \quad \bar{\theta}_j = \sup \{ \theta : (\theta, c) \in B(e_j) \}, \quad j = L, H.
$$

(17)
Consider a firm \( i \), some wages \( w_i \) and \( w_{-i} \), and a state \((\theta, c)\). Firm \( i \)'s maximum profit \( u_i^*(w_{-i}; \theta) \) is obtained by marginally outbidding \( w_{-i} \) when it is below \( \theta \), and by choosing the wage below \( w_{-i} \) and thus giving up the worker if \( \theta \leq w_{-i} \), so

\[
u_i^*(w_{-i}; \theta) = \sup_{w_i \geq 0} u_i(w_i, w_{-i}; \theta) = \max\{\theta - w_{-i}, 0\}.
\]

Observe that we only need to consider \( w_i \) and \( w_{-i} \) in \([\theta_j, \bar{\theta}_j]\). A wage above \( \bar{\theta}_j \) is dominated and cannot be a best compromise; a wage below \( \bar{\theta}_j \) will always be overbid by the rival's wage, as there is common knowledge that \( \theta \geq \theta_j \).

Suppose that \( w_i < w_{-i} \), so \( u_i(w_i, w_{-i}; \theta) = 0 \). Then the largest loss is obtained when \( \theta \) is the greatest:

\[
sup_{\theta; (\theta, c) \in B(e_j)} (u_i^*(w_{-i}; \theta) - u_i(w_i, w_{-i}; \theta)) \leq \max\{\bar{\theta}_j - w_{-i}, 0\}.
\]

Next, suppose that \( w_i > w_{-i} \), so \( u_i(w_i, w_{-i}; \theta) = \theta - w_i \). Then the largest loss is obtained when \( \theta \) is the smallest:

\[
sup_{\theta; (\theta, c) \in B(e_j)} (u_i^*(w_{-i}; \theta) - u_i(w_i, w_{-i}; \theta)) = \max\{\theta - w_{-i}, 0\} - (\theta - w_i) \leq w_i - \theta_j.
\]

Finally, suppose that \( w_i = w_{-i} \), so \( u_i(w_i, w_{-i}; \theta) = (\theta - w_i)/2 \). Then

\[
sup_{\theta; (\theta, c) \in B(e_j)} (u_i^*(w_{-i}; \theta) - u_i(w_i, w_{-i}; \theta)) = \max\{\theta - w_{-i}, 0\} - \frac{\theta - w_i}{2}
\]

\[
\leq \max\{0, \bar{\theta}_j - w_{-i}, (w_i - \theta_j)/2\}.
\]

The maximum loss \( l_i(w_i, w_{-i}) \) is given by the greatest of the three expressions, so

\[
l_i(w_i, w_{-i}) = \max\{0, \bar{\theta}_j - w_{-i}, w_i - \theta_j\}.
\]

The wages \( w_i \) that minimizes the maximum loss satisfies

\[
w_i = \bar{\theta}_j + \theta_j - w_{-i}, \quad i = 1, 2.
\]

So, we have obtained two equations, one for each \( i = 1, 2 \). Solving this pair of equations for \( w_1 \) and \( w_2 \) yields the best compromise \( w_i^*(e_j) \) for each firm \( i \), where

\[
w_i^*(e_j) = \frac{\bar{\theta}_j + \theta_j}{2}, \quad i = 1, 2.
\]

The associated maximum losses are

\[
l_i(w_i^*(e_j), w_{-i}^*(e_j)) = w_i^*(e_j) - \theta_j.
\]

Next, observe that the worker operates under complete information. Given each choice of \( e_j \), she anticipates the wages \( w_j = w_i^*(e_j) = w_2^*(e_j) \), \( j \in \{L, H\} \). So, given a state \((\theta, c)\), the worker chooses \( e = e_H \) if and only if

\[
w^H - c(\theta) \geq w^L.
\]

\[\text{The tie breaking is arbitrary, because the set of types is a continuum.}\]
Recall that $c(\theta)$ is strictly decreasing, and denote by $c^{-1}$ its inverse. Then, the worker chooses $e = e_H$ if and only if her type $\theta$ satisfies

$$\theta \geq c^{-1}(w^H - w^L).$$

**Pooling PCE.** If $w^H \leq w^L$, then every type chooses low level of education $e_L$, so the equilibrium is pooling. After observing $e = e_L$, the consistent conceivable set $B_L$ is thus the entire set of states, $B_L = \Omega$. By (7), the highest and lowest $\theta$ in $B_L$ are $\bar{\theta}_L = 1$ and $\theta_L = 0$. By (18), we obtain the equilibrium wages $w_i(e_L) = 1/2$. After observing an out-of-equilibrium education $e = e_H$, the conceivable set $B_H$ must induce the wage $w_i^*(e_H) \leq w_i^*(e_L)$. In particular, we can assume $B_H = \Omega$, and thus $w_i^*(e_H) = 1/2$.

Substituting the wage of $w_i^*(e) = 1/2$ and the lower bound productivity $\theta_L = 0$ into (19), we obtain the maximum loss for each firm $i$,

$$l_i(w_i^*(e_j), w_{-i}^*; e_j) = \frac{1}{2}, \quad i = 1, 2, \quad j = L, H.$$

**Separating PCE.** Consider now $w^H > w^L$, so that the worker with cost $c(\theta) \leq w^H - w^L$ chooses high education. Let

$$B_L = \{ (\theta, c) \in \Omega : c > w^H - w^L \} \quad \text{and} \quad B_H = \{ (\theta, c) \in \Omega : c(\theta) \leq w^H - w^L \}$$

be the conceivable sets of each firm when the level of education is $e_L$ and $e_H$, respectively. So, $B_L$ and $B_H$ contain all pairs $(\theta, c)$ such that low and high education is chosen, respectively. These sets thus satisfy the consistency requirement (Definition 1).

By (7) and (17), the highest and lowest $\theta$ in $B_H$ are given by

$$\bar{\theta}_H = 1 \quad \text{and} \quad \theta_H = c^{-1}(w^H - w^L) = \frac{1 - w^H + w^L}{b}. \quad (20)$$

Similarly, the highest and lowest $\theta$ in $B_L$ are given by

$$\bar{\theta}_L = c^{-1}(w^H - w^L) = \frac{1 + \delta - w^H + w^L}{b} \quad \text{and} \quad \theta_L = 0. \quad (21)$$

From (18), we have

$$w^H = \frac{\bar{\theta}_H + \theta_H}{2} \quad \text{and} \quad w^L = \frac{\bar{\theta}_L + \theta_L}{2}. \quad (22)$$

Solving the system of six equations in (20), (21), and (22), with six unknowns $(w^H, w^L, \bar{\theta}_H, \theta_H, \bar{\theta}_L, \theta_L)$, we obtain the equilibrium wages and the bounds on the productivity types as shown in (10) and (11).

Observe that the lowest possible cost of high education is $\operatorname{inf}\{c(\theta) : (\theta, c) \in \Omega\} = c^{-1}(1) = 1 - b$. Therefore, there exist states $(\theta, c)$ where high education $e_H$ is chosen if and only if $w^H - w^L > 1 - b$. Substituting our solution for $w^H$ and $w^L$ given by (10), we obtain that $w^H - w^L > 1 - b$ if and only if

$$\delta < 2b^2 - b.$$

This condition is thus necessary and sufficient for the existence of separating PCE.
Finally, substituting the wage $w^H$ and the productivity lower bound $\bar{\theta}_H$ into (19), we obtain firm $i$’s maximum loss when $e = e_H$,
\[ l_i(w^*_i(e_H), w^*_i(e_H); e_H) = w^H - \bar{\theta}_H = \frac{1}{2} - \frac{b + \delta}{4b^2}. \]
Substituting the wage $w^L$ and the productivity lower bound $\bar{\theta}_L$ into (19), we obtain the maximum loss when $e = e_L$,
\[ l_i(w^*_i(e_L), w^*_i(e_L); e_L) = w^L - \bar{\theta}_L = \frac{\delta}{2b} + \frac{b + \delta}{4b^2}. \]

A.5. Proof of Lemma 1. Let $p$ be an offered price and let $\alpha$ be an acceptance probability. The seller’s payoff in case of trade at price $p$ is $p - v$. If $p \geq v_0$, then this payoff is nonpositive for all $v \in [v_0, v_1]$. The seller can achieve the best payoff (and thus zero maximum loss) by rejecting the proposal, so $\alpha = 0$. Similarly, if $p \leq v_1$, then the seller’s payoff from accepting $p$ is nonnegative for all $v \in [v_0, v_1]$. The seller can achieve the best payoff (and thus zero maximum loss) by accepting the proposal, so $\alpha = 1$.

Suppose that $v_0 < p < v_1$. For each $v \in [v_0, p]$, the best payoff is $p - v$, so the loss from choosing $\alpha$ is
\[ (p - v) - \alpha(p - v) = (1 - \alpha)(p - v) \leq (1 - \alpha)(p - v_0). \]
For each $v \in [p, v_1]$, the best payoff is 0, so the loss from choosing $\alpha$ is
\[ 0 - \alpha(p - v) = \alpha(v - p) \leq \alpha(v_1 - p). \]
Thus, the maximum loss is
\[ l(p, \alpha; v) = \max\{(1 - \alpha)(p - v_0), \alpha(v_1 - p)\}. \]
The first term is decreasing and the second term is increasing in $\alpha$. The maximum loss is thus minimized by the solution of $(1 - \alpha)(p - v_0) = \alpha(v_1 - p)$, so $\alpha = (p - v_0)/(v_1 - v_0)$. \hfill \square

A.6. Proof of Proposition 4. Let us find a conceivable set and a strategy of the seller in the second stage for our PCE. The seller, who has observed $x$ and $p$ and has no information about $y \in [0, 1]$, has a conceivable set $B(x, p)$. Let $B(x, p)$ contain all states that cannot be ruled out given this information, so
\[ B(x, p) = \{(x', y') \in [0, 1]^2 : x' = x, y' \in [0, 1]\}. \]
So, $B(x, p)$ is consistent (Definition 1). Values are thus contained in
\[ \left\{ \frac{x + y}{2} : (x, y) \in B(x, p) \right\} = [x/2, (1 + x)/2]. \]
By Lemma 1, with $[v_0, v_1] = [x/2, (1 + x)/2]$, the seller’s acceptance strategy $\alpha^*(x, p)$ is given by (12).
We now find the buyer’s strategy in the first stage. The buyer’s maximum payoff given \((x, y)\) is

\[
\sup_{p' \in [0, 1]} \left( \frac{x + y}{2} - p' \right) \alpha^*(x, p') = \frac{y^2}{8}.
\]

Let \(p\) be a price. The buyer’s loss from choosing \(p\), given \((x, y)\), is given by

\[
\frac{y^2}{8} - \left( \frac{x + y}{2} - p \right) \alpha^*(x, p) \leq \max \left\{ \frac{1}{8} - \left( \frac{x + 1}{2} - p \right) \alpha^*(x, p), -\left( \frac{x}{2} - p \right) \alpha^*(x, p) \right\}
\]

\[
= \begin{cases} 
  \frac{1}{8}, & \text{if } x \geq 2p, \\
  \frac{1}{2}(2p - x)^2 + \max \left\{ \frac{1}{8} - p + \frac{x}{2}, 0 \right\}, & \text{if } 2p - 1 < x < 2p, \\
  p - \frac{x}{2}, & \text{if } x \leq 2p - 1.
\end{cases}
\]

(23)

The inequality follows from convexity of the loss in \(y\), so we can evaluate it at \(y = 1\) and \(y = 0\), and the equality is by the substitution of \(\alpha^*(x, p)\) from (12).

Observe that (23) is convex in \(x\) for \(x < 2p\) and constant for \(x \geq 2p\). So to maximize it w.r.t. \(x\), we only need to evaluate it at \(x = 0\) and \(x = 1\). It is then straightforward to see that \(p^* = 1/4\) minimizes the buyer’s maximum loss, which is equal to 1/8.

Finally, by Lemma 1, we can find the seller’s maximum loss. We thus find that the maximum losses of the buyer (proposer) and seller (responder) are

\[
l_b(p^*, \alpha^*) = \frac{1}{8} \quad \text{and} \quad \max_{x \in [0, 1]} l_s(p^*, \alpha^*; x) = \frac{1}{8}.
\]

\[\square\]

A.7. Proof of Proposition 5. Let us first determine a conceivable set and the best compromise acceptance strategy \(\alpha^*(p)\) for the buyer in the second stage in our PCE. The derivation is analogous Lemma 1. Specifically, let \(v \in [v_0, v_1]\). The buyer’s payoff in case of trade is \(v - p\). If \(p \geq v_1\), then the buyer’s payoff is nonpositive for all \(v \in [v_0, v_1]\). The buyer can achieve the best payoff (and thus the maximum loss equal to zero) by rejecting the proposal, so \(\alpha^*(p) = 0\). Similarly, if \(p \leq v_0\), then the buyer’s payoff is nonnegative for all \(v \in [v_0, v_1]\). The buyer can achieve the best payoff (and thus the maximum loss equal to zero) by accepting the proposal, so \(\alpha^*(p) = 1\).

Suppose that \(v_0 < p < v_1\). For each \(p \leq v\), the best payoff is \(v - p\), so the loss from accepting \(p\) with probability \(\alpha\) is

\[
(v - p) - \alpha(v - p) = (1 - \alpha)(v - p) \leq (1 - \alpha)(v_1 - p).
\]

For each \(p \geq v\), the best payoff is 0, so the loss from accepting \(p\) with probability \(\alpha\) is

\[
0 - \alpha(v - p) = \alpha(p - v) \leq \alpha(p - v_0).
\]

Thus, the maximum loss is

\[
l(p, \alpha; v) = \max\{(v_1 - \alpha)(1 - p), \alpha(p - v_0)\}.
\]

(24)
The first term is decreasing and the second term is increasing in $\alpha$. The maximum loss is thus minimized by the solution of $(1 - \alpha)(v_1 - p) = \alpha(p - v_0)$, so $\alpha^*(p) = (\bar{v} - p)/\bar{v}$.

The buyer knows that the seller chooses $p^* = 3/4$ for all $x \in [0, 1]$, so this price bears no information about $x$. Thus, the buyer who has observed $p^* = 3/4$ has a conceivable set that contains all states, so $B(p^*) = \Omega$. The interval of values is thus $[v_0, v_1] = [0, 1]$. The buyer’s best compromise acceptance probability is

$$\alpha^*(3/4) = \frac{v_1 - 3/4}{v_1 - v_0} = \frac{1}{4}.$$ 

Next, any price $p \neq 3/4$ is out of equilibrium. The buyer’s conceivable set in this case is set to $B(p) = \{(y, y) : y \in [0, 1]\}$. The interval of values is thus $[v_0, v_1] = [0, 1/2]$. Note that $p \geq v_0 = 0$. The buyer’s best compromise acceptance probability is

$$\alpha^*(p) = \max\left\{\frac{v_1 - p}{v_1 - v_0}, 0\right\} = \max\{1 - 2p, 0\}, \quad p \neq 3/4.$$ 

We now find the seller’s best compromise strategy in the first stage. Anticipating the buyer to play $\alpha^*$, the seller can obtain the following payoffs. Given $v = (x + y)/2$, the seller’s payoff is from $p^* = 3/4$ is

$$\hat{u}_s(v, p^*) = (p^* - v) \alpha^*(p^*) = \left(\frac{3}{4} - v\right) \frac{1}{4}.$$ 

The seller’s payoff is from $p \neq 3/4$ is

$$\hat{u}_s(v, p) = (p - v) \alpha^*(p) = \begin{cases} (p - v)(1 - 2p), & \text{if } p < 1/2, \\ 0, & \text{if } p \geq 1/2, p \neq 3/4. \end{cases}$$ 

The maximum payoff among all prices $p \neq 3/4$ is

$$\sup_{p \neq 3/4} \hat{u}_s(v, p) \geq \begin{cases} \frac{1}{8}(1 - 2v)^2, & \text{if } v < 1/2, \\ 0, & \text{if } v \geq 1/2. \end{cases}$$ 

The maximum loss of choosing $p \neq 3/4$ is thus greater than or equal to

$$\sup_{v \in [\frac{1}{2} + \epsilon, \frac{3}{2}]} (\hat{u}_s(v, p^*) - \hat{u}_s(v, p)) \geq \sup_{v \in [\frac{1}{2} + \epsilon, \frac{3}{2}]} \left(\frac{1}{8}(1 - 2v)^2 - \left(\frac{3}{4} - v\right) \frac{1}{4}\right) = \frac{3}{32} > \frac{1}{16}. $$

On the other hand, the maximum loss of choosing $p^* = 3/4$ satisfies

$$l_s(p^*, \alpha^*; x) = \sup_{v \in [\frac{1}{2} + \epsilon, \frac{3}{2}]} \left(\sup_{p \neq 3/4} \hat{u}_s(v, p) - \hat{u}_s(v, p^*)\right)$$

$$= \sup_{v \in [\frac{1}{2} + \epsilon, \frac{3}{2}]} \left(\frac{1}{8}(1 - 2v)^2 - \left(\frac{3}{4} - v\right) \frac{1}{4}\right).$$

(25)
It can be seen that the expression under the maximum in (25) is convex, so it needs to be evaluated at \( y = 0 \) and \( y = 1 \). So, for all \( x \in [0, 1] \),

\[
 l_s(p^*, a^*; x) = \max \left\{ \frac{2x^2 - 2x - 1}{16}, \frac{2x - 1}{16} \right\} = \frac{2x - 1}{16} \leq \frac{1}{16}.
\]

We thus conclude that choosing \( p^* = 3/4 \) gives a lower maximum loss than choosing \( p \neq 3/4 \), regardless of \( x \in [0, 1] \). Thus, \( p^* = 3/4 \) is a best compromise. The maximum losses of the seller (proposer) and buyer (responder) thus satisfy

\[
 \max_{x \in [0,1]} l_s(p^*, a^*; x) = \frac{1}{16} \quad \text{and} \quad l_b(p^*, a^*) = \frac{3}{16}. \quad \square
\]

References


In this appendix we analyze three examples that complement those presented in the main part of the paper.

B.1. Bilateral Trade with Simultaneous Offers. We continue the theme of bilateral trade in Section 3.4. This time, we consider the classic model of trade with private values and simultaneous moves (Chatterjee and Samuelson, 1983). A seller wants to sell an indivisible good to a buyer. The good has value $v_s$ for the seller and $v_b$ for the buyer. Each trader is privately informed about their own value, and it is commonly known that both values belong to $[0, 1]$.

Trade is organized using a double auction. The seller and the buyer simultaneously submit bids, $s$ and $b$, respectively, with $s, b \in [0, 1]$. If $s \leq b$, then the good is sold at the price $p = (s + b)/2$, so the seller and the buyer obtain the payoffs of $p - v_s$ and $v_b - p$, respectively. If $s > b$, then the good is not sold, and both parties obtain zero payoffs.

Let $s(v_s)$ and $b(v_b)$ denote the seller’s and the buyer’s strategies, respectively. We now find a perfect compromise equilibrium under the following regularity condition:

Strategies $s(v_s)$ and $b(v_b)$ are continuous and strictly increasing. \hfill (A_1)

We now present a PCE of this game.

**Proposition 6.** A pair of strategies $(s^*, b^*)$ that satisfies (A_1) is a PCE if and only if for all $v_s, v_b \in [0, 1]$

$$s^*(v_s) = \max \left\{ v_s, \frac{1}{4} + \frac{2v_s}{3} \right\} \quad \text{and} \quad b^*(v_b) = \min \left\{ v_b, \frac{1}{12} + \frac{2v_b}{3} \right\}. \quad (26)$$

The associated maximal losses are

$$l_s(v_s) = \max \left\{ \frac{1}{4} - \frac{v_s}{12(1 - v_s)}, 0 \right\} \leq \frac{1}{4} \quad \text{and} \quad l_b(v_b) = \max \left\{ \frac{1}{4} - \frac{1 - v_b}{12v_b}, 0 \right\} \leq \frac{1}{4}.$$

The PCE in Proposition 6 coincides with the PBE when $v_b$ and $v_s$ are uniformly distributed on $[0, 1]$ (Chatterjee and Samuelson, 1983).

**Proof.** Denote by $s^*(v_s)$ and $b^*(v_b)$ the strategies of the seller and the buyer in a best compromise equilibrium. Also, denote

$$s^* = \inf_{v_s \in [0, 1]} s^*(v_s) \quad \text{and} \quad b^* = \sup_{v_b \in [0, 1]} b^*(v_b).$$

Suppose that the seller’s value is $v_s$. Let us find the seller’s maximum loss for bidding $s$. If $v_s < b^*$, so there is a potential gain from trade when the buyer’s bid $b^*(v_b)$ is high enough, then there are two outcomes that the seller worries about. First, it is $s < b^*(v_b)$, so the seller could have made a greater bid, $s' = b^*(v_b)$, and sold the good at a higher price, thus obtaining a payoff increment of

$$\sup_{v_b} \left[ \left( b^*(v_b) - v_s \right) - \left( \frac{s + b^*(v_b)}{2} - v_s \right) \right] = \frac{b^* - s}{2}.$$
Second, it is \( s > b^*(v_b) > v_s \), so the good is not sold, but the seller could have made a smaller bid, \( s' = b^*(v_b) \), and sold the good, thus obtaining a payoff increment of
\[
\sup_{v_b} [b^*(v_b) - v_s] = s - v_s.
\]
The seller’s maximum loss is thus
\[
c_s(v_s, s, b^*) = \max\left\{ \frac{\bar{b}^* - s}{2}, s - v_s \right\} \text{ if } v_s < \bar{b}^*.
\]
The value of \( s \) that minimizes the above maximum loss is
\[
s^*(v_s) = \frac{\bar{b}^*}{3} + \frac{2v_s}{3}, \text{ if } v_s < \bar{b}^*.
\]
Alternatively, if there is no gain from trade, so \( v_s \geq \bar{b}^* \), then the seller obtains zero loss by choosing any bid that guarantees no trade, for example, \( s^*(v_s) = v_s \).

Symmetrically, we obtain that the buyer’s best compromise strategy satisfies
\[
b^*(v_b) = \begin{cases} 
\frac{s^*}{3} + \frac{2v_b}{3}, & \text{if } v_b > s^*, \\
v_b, & \text{if } v_b \leq s^*.
\end{cases}
\]
Since \( v_s, v_b \in [0, 1] \), we have
\[
s^* = s^*(0) = \frac{\bar{b}^*}{3} \text{ and } \bar{b}^* = b^*(1) = \frac{2 + s^*}{3}.
\]
Thus, \( s^* = 1/4 \) and \( \bar{b}^* = 3/4 \). Therefore, the PCE \((s^*, b^*)\) given by (26). \( \Box \)

B.2. Public Good Provision. Here we investigate the provision of a public good when each beneficiary knows her own value but not that of the others.

There are \( n \) agents, each has a private value \( v_i \in [0, \bar{v}] \) for the public good. Each agent \( i \) commits to contribute at most \( x_i \in [0, \bar{v}] \) in case the public good is provided. Agents make their commitments simultaneously.

The cost of providing the public good is \( c > 0 \). If the sum of committed contributions does not cover the cost, so \( \sum_{i=1}^{n} x_i < c \), then the public good is not provided, and each agent \( i \) obtains zero payoff. Otherwise, if \( \sum_{i=1}^{n} x_i \geq c \), then the public good is provided, and each agent \( i \) obtains the payoff
\[
v_i - t_i(x),
\]
where \( t_i(x) \) is the final transfer of agent \( i \) that depends on the profile of committed contributions \( x = (x_1, ..., x_n) \). We assume that the transfer rule \( t_i \) must satisfy:

(a) \( t_i(x) \leq x_i \), so no agent pays more than her committed contribution,
(b) \( \sum_{i=1}^{n} t_i(x) \geq c \),
(c) \( t = (t_1, ..., t_n) \) is symmetric, so agents are treated ex ante equally.

In addition, we assume that the cost of public good provision is relatively small, specifically,
\[
c \leq \frac{1}{2}(n - 1)\bar{v}.
\]
This assumption simplifies the exposition. The complementary case can also be easily analysed.
Let $s_i(v_i)$ be a strategy of agent $i$, so $x_i = s_i(v_i)$ specifies the maximal contribution of agent $i$. As in Section B.1, we restrict attention to strategies that satisfy the following assumption.

Strategies $s_i$ are continuous, strictly increasing, and $s_i(v) = s_j(v)$ for all $i, j = 1, \ldots, n$ and all $v \in [0, \bar{v}]$. (A2)

We compare three simple transfer rules.

(i) *Pay-as-you-bid rule.* Each agent pays as much as she commits to contribute whenever the good is provided, so

$$t_i(x) = x_i. \quad (28)$$

(ii) *Proportional rule.* Each agent $i$ pays proportionally to her commitment $x_i$ whenever the good is provided, so

$$t_i(x) = \frac{cx_i}{\sum_{j=1}^n x_j}. \quad (29)$$

(iii) *Additive rule.* Each agent $i$ pays the equal share $c/n$ plus the difference between her commitment and the average commitment, so

$$t_i(x) = \frac{c}{n} + x_i - \frac{1}{n} \sum_{j=1}^n x_j. \quad (30)$$

The assumptions (a), (b), and (c) are easily verified for these transfer rules.

We will measure the efficiency of a PCE profile $s^*$ by the ratio of the maximum welfare loss to the maximum possible surplus $n \bar{v}$. Our measure is denoted by $\bar{C}(s^*)$ and is given by

$$\bar{C}(s^*) = \sup_{(v_1, \ldots, v_n) \in [0, \bar{v}]^n} \left\{ \begin{array}{ll} \frac{1}{12n} \max\{0, \sum_i v_i - c\}, & \text{if } \sum_i s_i^*(v_i) \geq c, \\ 0, & \text{if } \sum_i s_i^*(v_i) < c. \end{array} \right.$$  

It turns out in the PCE presented below that the inefficiency emerges only when the public good is not provided when it is efficient to do so.

**Proposition 7.** A strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ that satisfies (A2) is a PCE if and only if for all $i = 1, \ldots, n$ and all $v_i \in [0, \bar{v}]$,

(i) if $t_i(x)$ is the pay-as-you-bid rule, then

$$s_i^*(v_i) = \frac{v_i}{2} \quad \text{and} \quad \bar{C}(s^*) = \frac{1}{2};$$

(ii) if $t_i(x)$ is the proportional rule, then

$$s_i^*(v_i) = \frac{v_i}{2} - c + \frac{1}{2} \sqrt{v_i^2 + 4c^2} \quad \text{and} \quad \bar{C}(s^*) = \frac{n}{2n + 1};$$

(iii) if $t_i(x)$ is the additive rule, then

$$s_i^*(v_i) = \frac{n}{2n - 1} v_i \quad \text{and} \quad \bar{C}(s^*) = \frac{n - 1}{2n - 1}.$$ 

Note that $\frac{1}{2} > \frac{n}{2n + 1} > \frac{n - 1}{2n - 1}$. So, the additive rule is the most efficient among these three transfer rules.
Proof. An agent who chooses \( x_i \) worries about two contingencies. First, it could be that the total contribution is just below \( c \), so \( \sum_j x_j = c - \varepsilon \), so the good is not provided, but had \( i \) contributed \( \varepsilon \) more it would have been provided. So, as \( \varepsilon \to 0 \), agent \( i \)'s loss is \( \max \{ v_i - x_i, 0 \} \).

Second, \( i \) could be that all other agents contribute enough to cover \( c \), so \( \sum_{j \neq i} x_j \geq c \), so the agent could have contributed nothing and still got the good. In this case the loss is the amount of contribution, \( t_i(x) \). Under all three assumptions, this loss is maximized when the other agents’ contributions exactly equal to the cost, so \( \sum_{j \neq i} x_j = c \). If \( t_i(x) \) is given by (28), then the loss is \( x_i \). If \( t_i(x) \) is given by (29), then the loss is \( c x_i/(c + x_i) \). If \( t_i(x) \) is given by (30), then the loss is \( (n-1)x_i/n \).

As the first type of loss is weakly decreasing and the second type of loss is strictly increasing in \( x_i \), we equalize the two and solve for \( x_i^*(v_i) \) that minimizes the maximum loss. For each of the assumptions about \( t_i(x) \), the solution \( x_i^*(v_i) \) is given by (i), (ii), and (iii) in Claim 7.

The maximum individual surplus is \( v_i \), so the maximal individual loss \( \varepsilon_i(v_i) \) is simply \( \max \{ v_i - x_i^*(v_i), 0 \} / v_i = 1 - x_i^*(v_i)/v_i \). Substituting the obtained solutions \( x_i^*(v_i) \) into this expression, we obtain \( \varepsilon_i(v_i) \) in (i), (ii), and (iii) in Claim 7.

It remains to verify that there exist values \( v_j \) for each \( j \neq i \) such that \( \sum_{j \neq i} x^*(v_j) \geq c \). If \( t_i(x) \) is given by (28), then

\[
\sup_{j \neq i} \sum x^*(v_j) = \sup_{j \neq i} \frac{v_j}{2} = \frac{(n-1)\bar{v}}{2},
\]

If \( t_i(x) \) is given by (29), then

\[
\sup_{j \neq i} \sum x^*(v_j) = \sup_{j \neq i} \left( \frac{v_j}{2} - c + \frac{1}{2}\sqrt{v_j^2 + 4c^2} \right)
= \left( \frac{n-1}{2} \right) \left( \bar{v} - c + \frac{1}{2}\sqrt{\bar{v}^2 + 4c^2} \right) \geq \frac{(n-1)\bar{v}}{2}.
\]

Finally, if \( t_i(x) \) is given by (30), then

\[
\sup_{j \neq i} \sum x^*(v_j) = \sup_{j \neq i} \frac{n v_j}{2n - 1} = (n - 1) \frac{n \bar{v}}{2n - 1} \geq \frac{(n-1)\bar{v}}{2}.
\]

Since \( c \leq \frac{(n-1)\bar{v}}{2} \) by assumption (27), we obtain that there exist values \( v_j \) for each \( j \neq i \) such that \( \sum_{j \neq i} x^*(v_j) \geq c \).

B.3. Forecasting. Here we consider the problem of forecasting of a variable whose underlying distribution is unknown. With this example we illustrate how noise influences learning. To keep the focus on learning, we present a one-shot decision problem of a single player.

Consider a forecaster who has to predict \( \theta \in [0,1] \). The variable \( \theta \) is drawn from a distribution \( F \) on \([0,1]\). The forecaster’s payoff is the quadratic loss:

\[ u(a, \theta) = -(a - \theta)^2. \]
Before making a prediction, the forecaster observes a noisy signal $z$ about $\theta$.

We analyze two variations of this model. In one variation, the forecaster knows how the noisy signal $z$ is generated but she is uncertain about the distribution of the fundamental variable $\theta$. In the other variation, the forecaster knows the distribution of $\theta$ but she is uncertain about the signal generating process.

### B.3.1. Unknown Distribution of Variable $\theta$

Here we are interested in how to make a prediction without knowing how the variable of interest is distributed.

Suppose that the forecaster does not know the distribution $F$. She only knows the expected value of this distribution, denoted by $\theta_0$. We allow for any such distribution $F$ that admits a density $f$ such that $\delta \leq f(\theta) \leq 1/\delta$ for some $\delta \in (0, 1)$. The last assumption excludes zero and unbounded densities. The set $\mathcal{F}_\delta$ of such distributions is thus given by

$$\mathcal{F}_\delta = \{ F \in \Delta([0, 1]) : E_F[\theta] = \theta_0 \text{ and } \delta \leq f(\theta) \leq 1/\delta \text{ for all } \theta \in [0, 1] \}.$$

The forecaster can condition her prediction on a noisy signal $z$ about $\theta$. The signal generating process is known and given by a conditional probability distribution $G(z|\theta)$. For a given $\varepsilon > 0$, signal $z$ reveals the true value $\theta$ with probability $1 - \varepsilon$ and a uniform draw from $[0, 1]$ with probability $\varepsilon$, so

$$G_\varepsilon(z|\theta) = \begin{cases} \varepsilon z, & \text{if } z < \theta, \\ 1 - \varepsilon + \varepsilon z, & \text{if } z \geq \theta. \end{cases}$$

Had the forecaster known the distribution $F \in \mathcal{F}_\delta$, she could have formed a posterior about $\theta$ conditional on the signal $z$. Let $E_{F,G_\varepsilon}[\cdot|z]$ denote the expectation under this posterior.

The maximum loss of a prediction $a \in [0, 1]$ given $z \in [0, 1]$ is

$$l(a; z) = \sup_{F \in \mathcal{F}_\delta} \left( \sup_{a' \in [0, 1]} E_{F,G_\varepsilon}[-(a' - \theta)^2|z] - E_{F,G_\varepsilon}[-(a - \theta)^2|z] \right).$$

A best compromise is a prediction $a^*(z)$ that achieves the least maximum loss, so

$$a^*(z) \in \arg \min_{a \in [0, 1]} l(a; z).$$

This problem can be embedded in our formal setting as described in Section 2. As the distribution $F$ is unknown, it is identified as the state. So the set of states is $\mathcal{F}_\delta$. In the formal game, first nature chooses a state $F$. Then the nonstrategic player $0$ observes $F$ and generates a signal $z$. Finally, the forecaster observes $z$ and makes a prediction.

**Proposition 8.** Let $\varepsilon \in [0, 1]$ and $\delta \in (0, 1)$. The forecaster’s best compromise is

$$a^*(z) = (1 - \lambda)z + \lambda \theta_0,$$

where

$$\lambda = \frac{\varepsilon}{2} \left( \frac{\delta}{1 - \varepsilon(1 - \delta)} + \frac{1}{\delta + \varepsilon(1 - \delta)} \right).$$
We postpone the proof to the end of this subsection. To describe the intuition, let us start with a simple observation.

**Lemma 2.** \( l(a; z) = \sup_{F \in \mathcal{F}_a} (a - E_{F,G_a}[\theta | z])^2. \)

The intuition is as follows. The variance of \( \theta \) conditional on a signal \( z \) enters the payoffs additively, and thus cancels out when computing the loss. As a result, the maximum loss \( l(a; z) \) is simply the maximum quadratic distance between \( a \) and the expected value of \( \theta \) conditional on \( z \).

**Proof of Lemma 2.** Fix \( G_z \). Let \( \bar{a}_F(z) = E_{F,G_a}[\theta | z] \). Observe that
\[
\bar{a}_F(z) \in \arg\max_{a' \in [0,1]} E_{F,G_a}[-(a' - \theta)^2 | z].
\] (31)

So, we have
\[
\sup_{a' \in [0,1]} E_{F,G_a}[-(a' - \theta)^2 | z] - E_{F,G_a}[-(a - \theta)^2 | z] = E_{F,G_a}[(a - \bar{a}_F(z)(a + \bar{a}_F(z) - 2\theta)) | z] = (a - \bar{a}_F(z))^2,
\]
where the first equality is by (31) and the last equality is by \( E_{F,G_a}[\theta | z] = \bar{a}_F(z) \). Thus,
\[
l(a; z) = \sup_{F \in \mathcal{F}_a} (a - \bar{a}_F(z))^2 = \sup_{F \in \mathcal{F}_a} (a - E_{F,G_a}[\theta | z])^2. \]

So, different distributions \( F \in \mathcal{F}_a \) induce different posterior means \( E_{F,G_a}[\theta | z] \). When making a prediction \( a \), the forecaster worries about a possible loss of \((a - E_{F,G_a}[\theta | z])^2 \). The best compromise is thus the midpoint between the highest posterior mean \( H(z) \) and the lowest posterior mean \( L(z) \) conditional on \( z \), where
\[
H(z) = \sup_{F \in \mathcal{F}_a} E_{F,G_a}[\theta | z] \quad \text{and} \quad L(z) = \inf_{F \in \mathcal{F}_a} E_{F,G_a}[\theta | z].
\] (32)

Note that the posterior mean \( E_{F,G_a}[\theta | z] \) is always between the prior mean \( \theta_0 \) and the observed signal \( z \). So, the best compromise \( a^*(z) \) can be expressed as a weighted average of \( z \) and \( \theta_0 \).

Note that the best compromise \( a^* \) is continuous in \( \varepsilon \) and satisfies
\[
a^*(z) \rightarrow z \quad \text{as} \quad \varepsilon \rightarrow 0 \quad \text{and} \quad a^*(z) \rightarrow \theta_0 \quad \text{as} \quad \varepsilon \rightarrow 1.
\]

As the noise vanishes, the signal becomes the best predictor. As the noise becomes dominant, so the signal becomes uninformative, the ex ante mean \( \theta_0 \) becomes the best predictor.

Now fix \( \varepsilon > 0 \) and observe that
\[
a^*(z) \rightarrow \frac{z + \theta_0}{2} \quad \text{as} \quad \delta \rightarrow 0.
\]

When the only assumption imposed on the set of distributions \( F \) is that \( E_F[\theta] = \theta_0 \), the best predictor is the midpoint between \( z \) and \( \theta_0 \). In particular, even though the noise \( \varepsilon \) may be very small, it is fixed and plays no role in this limit. Intuitively, depending on distribution \( F \), the signal \( z \) might be the best predictor. Or it might be useless when the value \( z \) of this signal under \( F \) is very unlikely, in which case
the signal contains hardly any information, and \( \theta_0 \) is the best predictor. The best compromise balances the losses in these two extreme cases.

Note the discontinuity of \( a^*(z) \) at \( \varepsilon = 0 \) and \( \delta = 0 \). This is because the limit of \( a^* \) depends on the order of the limits of \( \varepsilon \) and \( \delta \).

**Proof of Proposition 8.** Let \( H(z) \) and \( L(z) \) be the highest and lowest posterior means of \( \theta \) conditional on \( z \) given by (32). We have

\[
l(a; z) = \sup_{F \in F_a} \left( a - \mathbb{E}_{F,G_a}[\theta | z] \right)^2 = \max \left\{ (a - H(z))^2, (a - L(z))^2 \right\}
\]

where the first equality is by Lemma 2, and the last equality is by the convexity of the expression. Thus,

\[
a^*(z) = \inf_{a \in [0,1]} l(a; z) = \frac{1}{2} (H(z) + L(z)) .
\]

It remains to find \( H(z) \) and \( L(z) \). Suppose that \( z \geq \theta_0 \). Observe that

\[
\mathbb{E}_{F,G_a}[\theta | z] = \frac{(1 - \varepsilon) f(z) z + \varepsilon \int_0^1 \theta f(\theta) d\theta}{(1 - \varepsilon) f(z) + \varepsilon \int_0^1 f(\theta) d\theta} = \frac{(1 - \varepsilon) f(z) z + \varepsilon \theta_0}{(1 - \varepsilon) f(z) + \varepsilon}
\]

is increasing in \( f(z) \). Using the assumption that \( f(z) \leq 1/\delta \), we have

\[
H(z) = \sup_{F \in F_a} \left( \frac{(1 - \varepsilon)}{(1 - \varepsilon) f(z) + \varepsilon} \right) = \frac{(1 - \varepsilon) f(z) z + \varepsilon \theta_0}{(1 - \varepsilon) f(z) + \varepsilon} = \frac{(1 - \varepsilon) z + \varepsilon \delta \theta_0}{1 - \varepsilon + \varepsilon \delta} .
\]

Using the assumption that \( f(z) \geq \delta \), we have

\[
L(z) = \inf_{F \in F_a} \left( \frac{(1 - \varepsilon)}{(1 - \varepsilon) f(z) + \varepsilon} \right) = \frac{(1 - \varepsilon) f(z) z + \varepsilon \theta_0}{(1 - \varepsilon) f(z) + \varepsilon} = \frac{(1 - \varepsilon) \delta z + \varepsilon \theta_0}{(1 - \varepsilon) \delta + \varepsilon} .
\]

Analogously, for \( z \leq \theta_0 \) we obtain \( H(z) = \frac{(1 - \varepsilon) \delta z + \varepsilon \theta_0}{(1 - \varepsilon) \delta + \varepsilon} \) and \( L(z) = \frac{(1 - \varepsilon) \delta z + \varepsilon \theta_0}{(1 - \varepsilon) \delta + \varepsilon} \). Thus we obtain

\[
a^*(z) = \frac{1}{2} (H(z) + L(z)) = \frac{1}{2} \left( \frac{(1 - \varepsilon) z + \varepsilon \delta \theta_0}{1 - \varepsilon + \varepsilon \delta} + \frac{(1 - \varepsilon) \delta z + \varepsilon \theta_0}{(1 - \varepsilon) \delta + \varepsilon} \right) .
\]

\[ \square \]

**B.3.2. Unknown Distribution of Signal \( z \).** Here we are interested in how uncertain noise influences prediction.

Suppose that the forecaster knows the distribution \( F \) of \( \theta \), but is uncertain about how the signal \( z \) is generated. The signal generating process is given as follows. Let \( \delta > 0 \). Let the signal \( z \) be given by the sum of the variable \( \theta \) and a noise \( y \), so

\[
z = \theta + y ,
\]

where \( y \) is drawn independently from the interval \([-\delta, \delta]\). Note that \( z = \theta + y \in [-\delta, 1 + \delta] \), so \( z \) can be outside of \([0,1]\).

Let \( \varepsilon \in [0,1] \). With probability \( 1 - \varepsilon \), the noise \( y \) is drawn from a given distribution \( G_0 \) on \([-\delta, \delta]\). With the complementary probability \( \varepsilon \), the noise \( y \) is drawn from a distribution \( G \) over \([-\delta, \delta]\). We assume that the forecaster knows this process except that she does not know \( G \). We allow for all distributions \( G \) whose support is within \([-\delta, \delta]\). So the forecaster is fairly certain that the noise \( y \) is
drawn from $G_0$, but puts probability $\varepsilon$ that it is drawn from another distribution. Thus, a state is identified with the distribution $G$, and the set of states $\mathcal{G}_\delta$ is the set of all distributions on $[-\delta, \delta]$. Consequently, $\delta$ as the neighborhood size conditional on $z$ for a given $G \in \mathcal{G}_\delta$ this posterior. The maximum loss associated with a prediction $a \in [0, 1]$ given a signal $z \in [0, 1]$ is calculated as in Section B.3, except that now the set of states is $\mathcal{G}_\delta$, so

$$l(a; z) = \sup_{G \in \mathcal{G}_\delta} \left( \sup_{a' \in [0, 1]} \mathbb{E}_{F,G,\varepsilon}[-(a' - \theta)^2 | z] - \mathbb{E}_{F,G,\varepsilon}[-(a - \theta)^2 | z] \right).$$

In the next result, the distribution $F$ of $\theta$ is defined on $\mathbb{R}$ and its density is zero outside of $[0, 1]$.

**Proposition 9.** Let $\varepsilon \in [0, 1]$ and $\delta > 0$. The forecaster’s best compromise is

$$a^*(z) = \frac{1}{2} (H(z) + L(z)),$$

where

$$H(z) = \sup_{G \in \mathcal{G}_\delta} \mathbb{E}_{F,G,\varepsilon}[\theta | z] = \sup_{x \in [-\delta, \delta]} \frac{\varepsilon f(z - x)(z - x) + (1 - \varepsilon) \int_0^\delta (z - y) f(z - y) dG_0(y)}{\varepsilon f(z - x) + (1 - \varepsilon) \int_0^\delta f(z - y) dG_0(y)},$$

$$L(z) = \inf_{G \in \mathcal{G}_\delta} \mathbb{E}_{F,G,\varepsilon}[\theta | z] = \inf_{x \in [-\delta, \delta]} \frac{\varepsilon f(z - x)(z - x) + (1 - \varepsilon) \int_0^\delta (z - y) f(z - y) dG_0(y)}{\varepsilon f(z - x) + (1 - \varepsilon) \int_0^\delta f(z - y) dG_0(y)}.$$

The proof is analogous to that of Proposition 8 and thus omitted.

Just like in Section B.3, in this model the best compromise is the midpoint between the highest posterior mean $H(z)$ and the lowest posterior mean $L(z)$ conditional on $z$. The difference from Section B.3 is how these extreme posterior means are calculated. Observe that they are always in the $\delta$-neighborhood of $z$. As the neighborhood size $\delta$ approaches to 0, the extreme posterior means approach $z$, so

$$H(z) \to z \quad \text{and} \quad L(z) \to z \quad \text{as} \quad \delta \to 0.$$  

Consequently, $a^*(z) \to z$ as $\delta \to 0$.

Let is now fix $\delta > 0$ and vary the noise level $\varepsilon$. As the noise vanishes, $\varepsilon \to 0$, both extreme posterior means converge to the posterior mean under the benchmark distribution $G_0$, so

$$H(z) \to \mathbb{E}_{F,G_0,0}[\theta | z] \quad \text{and} \quad L(z) \to \mathbb{E}_{F,G_0,0}[\theta | z] \quad \text{as} \quad \varepsilon \to 0.$$  

Consequently, $a^*(z) \to \mathbb{E}_{F,G_0,0}[\theta | z]$ as $\varepsilon \to 0$. For instance, if $G_0$ is the uniform distribution, then the best predictor converges to the expected value of $\theta$ conditional on being within $\delta$ of the signal.

Finally, as $\varepsilon \to 1$, so the role of the benchmark $G_0$ disappears and any noise within $[-\delta, \delta]$ becomes possible. We obtain

$$H(z) \to z + \delta \quad \text{and} \quad L(z) \to z - \delta \quad \text{as} \quad \varepsilon \to 1.$$
Consequently, $a^*(z) \to z$ as $\varepsilon \to 1$. So, as the prior $G_0$ loses its value, the signal $z$ becomes the best predictor.